1　ゼロ点除去問題のメモ 1

$$\vec{x}_{k+1} = A_k\vec{x}_k + B_k\vec{w} \tag{1}$$

$$\vec{z}_k = H_k\vec{x}_k + \vec{v} \tag{2}$$

$$E[\vec{w}] = \vec{\mu} \neq \vec{0} \tag{3}$$

$$V[\vec{w}] \equiv E[(\vec{w}-\vec{\mu})(\vec{w}-\vec{\mu})^{\mathrm{T}}] = Q \tag{4}$$

$$E[\vec{v}] = 0 \tag{5}$$

$$V[\vec{v}] \equiv E[(\vec{v}-0)(\vec{v}-0)^{\mathrm{T}}] = R \tag{6}$$

$$E[\vec{w}_{modified}] \equiv \vec{0} \tag{7}$$

$$V[\vec{w}_{modified}] \equiv V[w] = Q \tag{8}$$

$$\vec{x}_{k+1} = A_k\vec{x}_k + B_k\vec{w} \equiv A_k\vec{x}_k + B_k\vec{\mu}_k + B_k\vec{w}_{modified} \tag{9}$$

$$\begin{bmatrix}\vec{x}\\ \vec{\mu}\end{bmatrix}_{k+1} = \begin{bmatrix}A_k & B_k\\ 0 & I\end{bmatrix}\begin{bmatrix}\vec{x}\\ \vec{\mu}\end{bmatrix}_k + \begin{bmatrix}B\\ 0\end{bmatrix}\vec{w}_{modified} \tag{10}$$

$$P_{extended} \equiv V[\begin{bmatrix}\vec{x}\\ \vec{\mu}\end{bmatrix}] = \begin{bmatrix}P & 0\\ 0 & S\end{bmatrix} \tag{11}$$

$$\begin{aligned}
P_{extended\ k+1} &= \begin{bmatrix}A_k & B_k\\ 0 & I\end{bmatrix} P_{extended\ k} \begin{bmatrix}A_k & B_k\\ 0 & I\end{bmatrix}^{\mathrm{T}} + \begin{bmatrix}B_k\\ 0\end{bmatrix} Q \begin{bmatrix}B_k\\ 0\end{bmatrix}^{\mathrm{T}}\\
&= \begin{bmatrix}A_k & B_k\\ 0 & I\end{bmatrix}\begin{bmatrix}P_k & 0\\ 0 & S\end{bmatrix}\begin{bmatrix}A_k^{\mathrm{T}} & 0\\ B_k^{\mathrm{T}} & I\end{bmatrix} + \begin{bmatrix}B_k\\ 0\end{bmatrix} Q \begin{bmatrix}B_k^{\mathrm{T}} & 0\end{bmatrix}\\
&= \begin{bmatrix}AP_kA^{\mathrm{T}} + B_kSB_k^{\mathrm{T}} & B_kS\\ SB_k^{\mathrm{T}} & S\end{bmatrix} + \begin{bmatrix}B_kQB_k^{\mathrm{T}} & 0\\ 0 & 0\end{bmatrix}\\
&= \begin{bmatrix}P_{k+1} + B_kSB_k^{\mathrm{T}} & B_kS\\ SB_k^{\mathrm{T}} & S\end{bmatrix}
\end{aligned} \tag{12}$$

$$\vec{z}_k = \begin{bmatrix}H & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\vec{x}\\ \vec{\mu}\end{bmatrix}_k + \begin{bmatrix}I\\ 0\end{bmatrix}\vec{v} \tag{13}$$

2 ゼロ点除去問題のメモ 2

Kalman Filter においてモデル化されていないバイアス成分 $\underline{\mu}$ が混入していた場合、すなわち

$$\overline{\underline{x}}_{k+1} = \Phi_k \hat{\underline{x}}_k + \Gamma_k \left(\underline{\mu}_k + \underline{u}_{ideal\,k} \right) \tag{14}$$

における影響を調べる。

定義は以下のとおり。

$$E\left[\underline{\mu}_k\right] \approx \underline{\mu}_k \tag{15}$$

$$V\left[\underline{\mu}_k\right] \approx \underline{0} \tag{16}$$

$$E\left[\underline{\mu}_k \underline{u}_{ideal\,k}^{\mathrm{T}}\right] = \underline{0} \tag{17}$$

$$V\left[\underline{\mu}_k + \underline{u}_{ideal\,k}\right] = V\left[\underline{u}_{ideal\,k}\right] = Q \tag{18}$$

$$\overline{\underline{x}}_{no\,bias\,k+1} \equiv \Phi_k \hat{\underline{x}}_k + \Gamma_k \underline{u}_{ideal\,k} \tag{19}$$

$$\hat{\underline{x}}_{ideal\,0} \equiv \hat{\underline{x}}_0 \tag{20}$$

$$\hat{\underline{x}}_{ideal\,k+1} \equiv \left(\Phi_k \hat{\underline{x}}_{ideal\,k} + \Gamma_k \underline{u}_{ideal\,k}\right) + K_{k+1}\left(\underline{z}_{k+1} - H_{k+1}\left(\Phi_k \hat{\underline{x}}_{ideal\,k} + \Gamma_k \underline{u}_{ideal\,k}\right)\right) \tag{21}$$

$$\overline{\underline{x}}_{ideal\,k+1} \equiv \Phi_k \hat{\underline{x}}_{ideal\,k} + \Gamma_k \underline{u}_{ideal\,k} \tag{22}$$

$$E\left[\underline{z}_{k+1} - H_{k+1}\overline{\underline{x}}_{ideal\,k+1}\right] = \underline{0} \tag{23}$$

$$V\left[\underline{z}_{k+1} - H_{k+1}\overline{\underline{x}}_{ideal\,k+1}\right] = R \tag{24}$$

これらの定義から

$$\begin{aligned}
\hat{\underline{x}}_{k+1} &= \overline{\underline{x}}_{k+1} + K_{k+1}\left(\underline{z}_{k+1} - H_{k+1}\overline{\underline{x}}_{k+1}\right) \\
&= \left(\Phi_k \hat{\underline{x}}_k + \Gamma_k\left(\underline{\mu}_k + \underline{u}_{ideal\,k}\right)\right) + K_{k+1}\left(\underline{z}_{k+1} - H_{k+1}\left(\Phi_k \hat{\underline{x}}_k + \Gamma_k\left(\underline{\mu}_k + \underline{u}_{ideal\,k}\right)\right)\right) \\
&= \left(\Phi \hat{\underline{x}}_k + \Gamma_k \underline{u}_{ideal\,k}\right) + K_{k+1}\left(\underline{z}_{k+1} - H_{k+1}\left(\Phi_k \hat{\underline{x}}_k + \Gamma_k \underline{u}_{ideal\,k}\right)\right) + \left(I - K_{k+1}H_{k+1}\right)\Gamma_k \underline{\mu}_k \\
&= \overline{\underline{x}}_{no\,bias\,k+1} + K_{k+1}\left(\underline{z}_{k+1} - H_{k+1}\overline{\underline{x}}_{no\,bias\,k+1}\right) + \left(I - K_{k+1}H_{k+1}\right)\Gamma_k \underline{\mu}_k
\end{aligned} \tag{25}$$

ここで以下の $AdditionalTerm.$ を求める。

$$\begin{aligned}
\overline{\underline{x}}_{no\,bias\,k+1} &= \overline{\underline{x}}_{ideal\,k+1} + Additional\ Term. \\
&= \Phi_k \hat{\underline{x}}_{ideal\,k} + \Gamma_k \underline{u}_{ideal\,k} + Additional\ Term.
\end{aligned} \tag{26}$$

順々に計算をしてみる。

$$\overline{\underline{x}}_{no\,bias\,1} = \Phi_0 \hat{\underline{x}}_{ideal\,0} + \Gamma_0 \underline{u}_{ideal\,0} \tag{27}$$

$$\begin{aligned}
\hat{\underline{x}}_1 &= \overline{\underline{x}}_{no\,bias\,1} + K_1\left(\underline{z}_1 - H_1\overline{\underline{x}}_{no\,bias\,1}\right) + \left(I - K_1H_1\right)\Gamma_0 \underline{\mu}_0 \\
&= \hat{\underline{x}}_{ideal\,1} + \left(I - K_1H_1\right)\Gamma_0 \underline{\mu}_0
\end{aligned} \tag{28}$$

$$
\begin{aligned}
\underline{\bar{x}}_{no\,bias\,2} &= \Phi_1 \underline{\hat{x}}_1 + \Gamma_1 \underline{u}_{ideal\,1} \\
&= \Phi_1 \underline{\hat{x}}_{ideal\,1} + \Gamma_1 \underline{u}_{ideal\,1} + \Phi_1 \left(I - K_1 H_1\right) \Gamma_0 \underline{\mu}_0 \\
&= \underline{\bar{x}}_{ideal\,2} + \Phi_1 \left(I - K_1 H_1\right) \Gamma_0 \underline{\mu}_0
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
\underline{\hat{x}}_2 &= \underline{\bar{x}}_{no\,bias\,2} + K_2 \left(\underline{z}_2 - H_2 \underline{\bar{x}}_{no\,bias\,2}\right) + \left(I - K_2 H_2\right) \Gamma_1 \underline{\mu}_1 \\
&= \underline{\hat{x}}_{ideal\,2} + \left(I - K_2 H_2\right) \Big(\Phi_1 \left(I - K_1 H_1\right) \Gamma_0 \underline{\mu}_0 + \Gamma_1 \underline{\mu}_1 \Big)
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
\underline{\bar{x}}_{no\,bias\,3} &= \Phi_2 \underline{\hat{x}}_2 + \Gamma_2 \underline{u}_{ideal\,2} \\
&= \Phi_2 \underline{\hat{x}}_{ideal\,2} + \Gamma_2 \underline{u}_{ideal\,2} \\
&\quad + \Phi_2 \left(I - K_2 H_2\right) \Big(\Phi_1 (I - K_1 H_1) \Gamma_0 \underline{\mu}_0 + \Gamma_1 \underline{\mu}_1 \Big) \\
&= \underline{\bar{x}}_{ideal\,3} + \sum_{i=0}^{1} \left(\prod_{j=2,\,j--}^{i+1} \Phi_j \left(I - K_j H_j\right) \right) \Gamma_i \underline{\mu}_i
\end{aligned} \tag{31}
$$

以上の計算結果から

$$
\underline{\bar{x}}_{no\,bias\,k+1} = \underline{\bar{x}}_{ideal\,k+1} + \underbrace{\sum_{i=0}^{k-1} \left(\prod_{j=k,\,j--}^{i+1} \Phi_j \left(I - K_j H_j\right) \right) \Gamma_i \underline{\mu}_i}_{Additional\,Term.} \tag{32}
$$

これより以下の式が導かれる。

$$
\begin{aligned}
\underline{\bar{x}}_{k+1} &= \underline{\bar{x}}_{no\,bias\,k+1} + \Gamma_k \underline{\mu}_k \\
&= \underline{\bar{x}}_{ideal\,k+1} + \sum_{i=0}^{k-1} \left(\prod_{j=k,\,j--}^{i+1} \Phi_j \left(I - K_j H_j\right) \right) \Gamma_i \underline{\mu}_i + \Gamma_k \underline{\mu}_k \\
&= \underline{\bar{x}}_{ideal\,k+1} + \sum_{i=0}^{k} \left(\prod_{j=k,\,j--}^{i+1} \Phi_j \left(I - K_j H_j\right) \right) \Gamma_i \underline{\mu}_i \qquad \left(\because \prod_{j=k,\,j--}^{k+1} = I \right)
\end{aligned} \tag{33}
$$

$$
\begin{aligned}
\underline{\hat{x}}_{k+1} &= \underline{\bar{x}}_{k+1} + K_{k+1} \left(\underline{z}_{k+1} - H_{k+1} \underline{\bar{x}}_{k+1}\right) \\
&= \underline{\bar{x}}_{ideal\,k+1} + K_{k+1} \left(\underline{z}_{k+1} - H_{k+1} \underline{\bar{x}}_{ideal\,k+1}\right) \\
&\quad + \left(I - K_{k+1} H_{k+1}\right) \sum_{i=0}^{k} \left(\prod_{j=k,\,j--}^{i+1} \Phi_j \left(I - K_j H_j\right) \right) \Gamma_i \underline{\mu}_i \\
&= \underline{\hat{x}}_{ideal\,k+1} + \left(I - K_{k+1} H_{k+1}\right) \sum_{i=0}^{k} \left(\prod_{j=k,\,j--}^{i+1} \Phi_j \left(I - K_j H_j\right) \right) \Gamma_i \underline{\mu}_i
\end{aligned} \tag{34}
$$

$$
\underline{\hat{x}}_{ideal\,k+1} = \underline{\hat{x}}_{k+1} - \left(I - K_{k+1} H_{k+1}\right) \sum_{i=0}^{k} \left(\prod_{j=k,\,j--}^{i+1} \Phi_j \left(I - K_j H_j\right) \right) \Gamma_i \underline{\mu}_i \tag{35}
$$

また

$$
\begin{aligned}
E\left[\underline{z}_{k+1}-H_{k+1}\overline{\underline{x}}_{k+1}\right] &= E\left[\underline{z}_{k+1}-H_{k+1}\left(\overline{\underline{x}}_{ideal\,k+1}+\sum_{i=0}^{k}\left(\prod_{j=k,\,j--}^{i+1}\Phi_j\left(I-K_jH_j\right)\right)\Gamma_i\underline{\mu}_i\right)\right]\\
&= E\left[\underline{z}_{k+1}-H_{k+1}\overline{\underline{x}}_{ideal\,k+1}\right]\\
&\quad -E\left[H_{k+1}\left(\sum_{i=0}^{k}\left(\prod_{j=k,\,j--}^{i+1}\Phi_j\left(I-K_jH_j\right)\right)\Gamma_i\underline{\mu}_i\right)\right]\\
&= -H_{k+1}\sum_{i=0}^{k}\left(\prod_{j=k,\,j--}^{i+1}\Phi_j\left(I-K_jH_j\right)\right)\Gamma_i\ E\left[\underline{\mu}_i\right]\\
&\approx -H_{k+1}\sum_{i=0}^{k}\left(\prod_{j=k,\,j--}^{i+1}\Phi_j\left(I-K_jH_j\right)\right)\Gamma_i\underline{\mu}_i
\end{aligned}
\tag{36}
$$

ここで

$$\underline{\mu}_i=\underline{\mu}=const.\quad (0\leqq i\leqq k) \tag{37}$$

とすれば

$$E\left[\underline{z}_{k+1}-H_{k+1}\overline{\underline{x}}_{k+1}\right]=-\left\{H_{k+1}\sum_{i=0}^{k}\left(\prod_{j=k,\,j--}^{i+1}\Phi_j\left(I-K_jH_j\right)\right)\Gamma_i\right\}\underline{\mu} \tag{38}$$

さらに

$$E\left[\underline{z}_{k+1}-H_{k+1}\overline{\underline{x}}_{k+1}\right]\approx\frac{1}{k}\sum_{i=0}^{k}\left(\underline{z}_{i+1}-H_{i+1}\overline{\underline{x}}_{i+1}\right) \tag{39}$$

従ってバイアス成分の推定は最小二乗法を用いて

$$
\begin{aligned}
\underline{\mu} &\approx -\left(\{\}^{\mathrm{T}}\{\}\right)^{-1}\{\}^{\mathrm{T}}E\left[\underline{z}_{k+1}-H_{k+1}\overline{\underline{x}}_{k+1}\right]\\
&\approx -\left(\{\}^{\mathrm{T}}\{\}\right)^{-1}\{\}^{\mathrm{T}}\frac{1}{k}\sum_{i=0}^{k}\left(\underline{z}_{i+1}-H_{i+1}\overline{\underline{x}}_{i+1}\right)
\end{aligned}
\tag{40}
$$

で $\overline{\underline{x}}_i\,(1\leqq i\leqq k+1)$ の計算の際に含まれていたバイアス成分 $\underline{\mu}$ を推定し、それによって影響をうけた分を式 (35) で補正をおこなえばよい。以後 ($k+2\leqq i$) は $\underline{\mu}$ を引いた形、例えば

$$\overline{\underline{x}}_{k+2}=\Phi_{k+1}\hat{\underline{x}}_{k+1}+\Gamma_{k+1}\left(\underline{\mu}_{k+1}+\underline{u}_{ideal\,k+1}-\underline{\mu}\right) \tag{41}$$

で計算を行い、 $i=n(k+1)$ で推定、補正を繰り返せばよい。

k は、バイアス成分の性質 (非常にゆっくりとした変化である) を考慮して大きめに取るとよいだろう。(高周波カットの性質を持つようになる、このあたりのことは平均化フィルタを z 変換とかで考察してみるとわかるはず)