1　線形化に関して

- Quarternion 演算の線形化

$$
\begin{aligned}
\Delta\tilde{q}_1^*\tilde{q}_2\tilde{q}_1 &= \begin{Bmatrix} \Delta q_{10} \\ -\Delta\boldsymbol{q}_1 \end{Bmatrix} \begin{Bmatrix} q_{20} \\ \boldsymbol{q}_2 \end{Bmatrix} \begin{Bmatrix} q_{10} \\ \boldsymbol{q}_1 \end{Bmatrix} \\
&= \begin{Bmatrix} q_{20}\Delta q_{10} + \boldsymbol{q}_2\cdot\Delta\boldsymbol{q}_1 \\ \boldsymbol{q}_2\Delta q_{10} - q_{20}\Delta\boldsymbol{q}_1 + \boldsymbol{q}_2\times\Delta\boldsymbol{q}_1 \end{Bmatrix} \begin{Bmatrix} q_{10} \\ \boldsymbol{q}_1 \end{Bmatrix} \\
&= \begin{Bmatrix} (q_{10}q_{20}\Delta q_{10} + q_{10}\boldsymbol{q}_2\cdot\Delta\boldsymbol{q}_1) \\ -((\boldsymbol{q}_1\cdot\boldsymbol{q}_2)\Delta q_{10} - q_{20}\boldsymbol{q}_1\cdot\Delta\boldsymbol{q}_1 - \boldsymbol{q}_2\times\boldsymbol{q}_1\cdot\Delta\boldsymbol{q}_1) \\ (q_{20}\boldsymbol{q}_1\Delta q_{10} + (\boldsymbol{q}_2\cdot\Delta\boldsymbol{q}_1)\boldsymbol{q}_1) + (q_{10}\boldsymbol{q}_2\Delta q_{10} - q_{10}q_{20}\Delta\boldsymbol{q}_1 + q_{10}\boldsymbol{q}_2\times\Delta\boldsymbol{q}_1) \\ +(\boldsymbol{q}_2\times\boldsymbol{q}_1\Delta q_{10} + q_{20}\boldsymbol{q}_1\times\Delta\boldsymbol{q}_1 + (\boldsymbol{q}_2\times\Delta\boldsymbol{q}_1)\times\boldsymbol{q}_1) \end{Bmatrix} \\
&= \begin{Bmatrix} (q_{10}q_{20} - \boldsymbol{q}_1\cdot\boldsymbol{q}_2)\Delta q_{10} + (q_{10}\boldsymbol{q}_2 + q_{20}\boldsymbol{q}_1 + \boldsymbol{q}_2\times\boldsymbol{q}_1)\cdot\Delta\boldsymbol{q}_1 \\ (q_{10}\boldsymbol{q}_2 + q_{20}\boldsymbol{q}_2 + \boldsymbol{q}_2\times\boldsymbol{q}_1)\Delta q_{10} - (q_{10}q_{20} - \boldsymbol{q}_1\cdot\boldsymbol{q}_2)\Delta\boldsymbol{q}_1 + (q_{10}\boldsymbol{q}_2 + q_{20}\boldsymbol{q}_2)\times\Delta\boldsymbol{q}_1 - (\boldsymbol{q}_1\cdot\Delta\boldsymbol{q}_1)\boldsymbol{q}_2 + (\boldsymbol{q}_2\cdot\Delta\boldsymbol{q}_1)\boldsymbol{q}_1 \end{Bmatrix} \\
&= \begin{Bmatrix} (q_{10}q_{20} - \boldsymbol{q}_1\cdot\boldsymbol{q}_2)\Delta q_{10} + (q_{10}\boldsymbol{q}_2 + q_{20}\boldsymbol{q}_1 + \boldsymbol{q}_2\times\boldsymbol{q}_1)\cdot\Delta\boldsymbol{q}_1 \\ (q_{10}\boldsymbol{q}_2 + q_{20}\boldsymbol{q}_2 + \boldsymbol{q}_2\times\boldsymbol{q}_1)\Delta q_{10} - (q_{10}q_{20} - \boldsymbol{q}_1\cdot\boldsymbol{q}_2)\Delta\boldsymbol{q}_1 + (q_{10}\boldsymbol{q}_2 + q_{20}\boldsymbol{q}_2 + \boldsymbol{q}_2\times\boldsymbol{q}_1)\times\Delta\boldsymbol{q}_1 \end{Bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{q}_1^*\tilde{q}_2\Delta\tilde{q}_1 &= \begin{Bmatrix} q_{10} \\ -\boldsymbol{q}_1 \end{Bmatrix} \begin{Bmatrix} q_{20} \\ \boldsymbol{q}_2 \end{Bmatrix} \begin{Bmatrix} \Delta q_{10} \\ \Delta\boldsymbol{q}_1 \end{Bmatrix} \\
&= \begin{Bmatrix} q_{10}q_{20} + \boldsymbol{q}_1\cdot\boldsymbol{q}_2 \\ q_{10}\boldsymbol{q}_2 - q_{20}\boldsymbol{q}_1 + \boldsymbol{q}_2\times\boldsymbol{q}_1 \end{Bmatrix} \begin{Bmatrix} \Delta q_{10} \\ \Delta\boldsymbol{q}_1 \end{Bmatrix} \\
&= \begin{Bmatrix} (q_{10}q_{20} + \boldsymbol{q}_1\cdot\boldsymbol{q}_2)\Delta q_{10} - (q_{10}\boldsymbol{q}_2 - q_{20}\boldsymbol{q}_1 + \boldsymbol{q}_2\times\boldsymbol{q}_1)\cdot\Delta\boldsymbol{q}_1 \\ (q_{10}q_{20} + \boldsymbol{q}_1\cdot\boldsymbol{q}_2)\Delta\boldsymbol{q}_1 + (q_{10}\boldsymbol{q}_2 - q_{20}\boldsymbol{q}_1 + \boldsymbol{q}_2\times\boldsymbol{q}_1)\Delta q_{10} + (q_{10}\boldsymbol{q}_2 - q_{20}\boldsymbol{q}_1 + \boldsymbol{q}_2\times\boldsymbol{q}_1)\times\Delta\boldsymbol{q}_1 \end{Bmatrix} \\
&= \begin{Bmatrix} (q_{10}q_{20} + \boldsymbol{q}_1\cdot\boldsymbol{q}_2)\Delta q_{10} - (q_{10}\boldsymbol{q}_2 - q_{20}\boldsymbol{q}_1 + \boldsymbol{q}_2\times\boldsymbol{q}_1)\cdot\Delta\boldsymbol{q}_1 \\ (q_{10}\boldsymbol{q}_2 - q_{20}\boldsymbol{q}_2 + \boldsymbol{q}_2\times\boldsymbol{q}_1)\Delta q_{10} + (q_{10}q_{20} + \boldsymbol{q}_1\cdot\boldsymbol{q}_2)\Delta\boldsymbol{q}_1 + (q_{10}\boldsymbol{q}_2 - q_{20}\boldsymbol{q}_2 + \boldsymbol{q}_2\times\boldsymbol{q}_1)\times\Delta\boldsymbol{q}_1 \end{Bmatrix}
\end{aligned}
$$

$$
\Delta\tilde{q}_1^* \begin{Bmatrix} 0 \\ \boldsymbol{v} \end{Bmatrix} \tilde{q}_1 = \begin{Bmatrix} (-\boldsymbol{q}_1\cdot\boldsymbol{v})\Delta q_{10} + (q_{10}\boldsymbol{v} + \boldsymbol{v}\times\boldsymbol{q}_1)\cdot\Delta\boldsymbol{q}_1 \\ (q_{10}\boldsymbol{v} + \boldsymbol{v}\times\boldsymbol{q}_1)\Delta q_{10} + (\boldsymbol{q}_1\cdot\boldsymbol{v})\Delta\boldsymbol{q}_1 + (q_{10}\boldsymbol{v} + \boldsymbol{v}\times\boldsymbol{q}_1)\times\Delta\boldsymbol{q}_1 \end{Bmatrix}
$$

$$
\tilde{q}_1^* \begin{Bmatrix} 0 \\ \boldsymbol{v} \end{Bmatrix} \Delta\tilde{q}_1 = \begin{Bmatrix} (\boldsymbol{q}_1\cdot\boldsymbol{v})\Delta q_{10} - (q_{10}\boldsymbol{v} + \boldsymbol{v}\times\boldsymbol{q}_1)\cdot\Delta\boldsymbol{q}_1 \\ (q_{10}\boldsymbol{v} + \boldsymbol{v}\times\boldsymbol{q}_1)\Delta q_{10} + (\boldsymbol{q}_1\cdot\boldsymbol{v})\Delta\boldsymbol{q}_1 + (q_{10}\boldsymbol{v} + \boldsymbol{v}\times\boldsymbol{q}_1)\times\Delta\boldsymbol{q}_1 \end{Bmatrix}
$$

$$
\begin{aligned}
\Delta\tilde{q}_1^* \begin{Bmatrix} 0 \\ \boldsymbol{v} \end{Bmatrix} \tilde{q}_1 + \tilde{q}_1^* \begin{Bmatrix} 0 \\ \boldsymbol{v} \end{Bmatrix} \Delta\tilde{q}_1 &= 2 \begin{Bmatrix} 0 \\ (q_{10}\boldsymbol{v} + \boldsymbol{v}\times\boldsymbol{q}_1)\Delta q_{10} + (\boldsymbol{q}_1\cdot\boldsymbol{v})\Delta\boldsymbol{q}_1 + (q_{10}\boldsymbol{v} + \boldsymbol{v}\times\boldsymbol{q}_1)\times\Delta\boldsymbol{q}_1 \end{Bmatrix} \\
&= 2 \left\{ \begin{matrix} 0 \\ \begin{bmatrix} \omega_1 & \lambda & -\omega_3 & \omega_2 \\ \omega_2 & \omega_3 & \lambda & -\omega_1 \\ \omega_3 & -\omega_2 & \omega_1 & \lambda \end{bmatrix} \quad \begin{matrix} \lambda = \boldsymbol{q}_1\cdot\boldsymbol{v} \\ \omega = q_{10}\boldsymbol{v} + \boldsymbol{v}\times\boldsymbol{q}_1 \end{matrix} \quad \Delta \begin{bmatrix} q_{10} \\ \boldsymbol{q}_1 \end{bmatrix} \end{matrix} \right\}
\end{aligned}
$$

$$
q_{10}\boldsymbol{v} + \boldsymbol{v}\times\boldsymbol{q}_1 = \begin{bmatrix} v_1 & 0 & -v_3 & v_2 \\ v_2 & v_3 & 0 & -v_1 \\ v_3 & -v_2 & v_1 & 0 \end{bmatrix} \begin{bmatrix} q_{10} \\ \boldsymbol{q}_1 \end{bmatrix}
$$

$$\Delta\tilde{q}_1 \begin{Bmatrix} 0 \\ \boldsymbol{v} \end{Bmatrix} \tilde{q}_1^* + \tilde{q}_1 \begin{Bmatrix} 0 \\ \boldsymbol{v} \end{Bmatrix} \Delta\tilde{q}_1^* = 2 \begin{Bmatrix} 0 \\ (q_{10}\boldsymbol{v} - \boldsymbol{v}\times\boldsymbol{q}_1)\Delta q_{10} + (\boldsymbol{q}_1\cdot\boldsymbol{v})\Delta\boldsymbol{q}_1 - (q_{10}\boldsymbol{v} - \boldsymbol{v}\times\boldsymbol{q}_1)\times\Delta\boldsymbol{q}_1 \end{Bmatrix}$$

$$= 2 \left\{ \begin{matrix} 0 \\ \begin{bmatrix} \omega_1 & \lambda & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & \lambda & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & \lambda \end{bmatrix} \Delta \begin{bmatrix} q_{10} \\ \boldsymbol{q}_1 \end{bmatrix} \\ \lambda = \boldsymbol{q}_1\cdot\boldsymbol{v} \\ \omega = q_{10}\boldsymbol{v} - \boldsymbol{v}\times\boldsymbol{q}_1 \end{matrix} \right\}$$

$$q_{10}\boldsymbol{v} - \boldsymbol{v}\times\boldsymbol{q}_1 = \begin{bmatrix} v_1 & 0 & v_3 & -v_2 \\ v_2 & -v_3 & 0 & v_1 \\ v_3 & v_2 & -v_1 & 0 \end{bmatrix} \begin{bmatrix} q_{10} \\ \boldsymbol{q}_1 \end{bmatrix}$$

$$\begin{aligned} \tilde{q}^* \begin{Bmatrix} 0 \\ \Delta\boldsymbol{v} \end{Bmatrix} \tilde{q} &= \begin{Bmatrix} q_0 \\ -\boldsymbol{q} \end{Bmatrix} \begin{Bmatrix} 0 \\ \Delta\boldsymbol{v} \end{Bmatrix} \begin{Bmatrix} q_0 \\ \boldsymbol{q} \end{Bmatrix} \\ &= \begin{Bmatrix} \boldsymbol{q}\cdot\Delta\boldsymbol{v} \\ q_0\Delta\boldsymbol{v} - \boldsymbol{q}\times\Delta\boldsymbol{v} \end{Bmatrix} \begin{Bmatrix} q_0 \\ \boldsymbol{q} \end{Bmatrix} \\ &= \begin{Bmatrix} q_0\boldsymbol{q}\cdot\Delta\boldsymbol{v} - (q_0\Delta\boldsymbol{v}\cdot\boldsymbol{q} - (\boldsymbol{q}\times\Delta\boldsymbol{v})\cdot\boldsymbol{q}) \\ (\boldsymbol{q}\cdot\Delta\boldsymbol{v})\boldsymbol{q} + (q_0^2\Delta\boldsymbol{v} - q_0\boldsymbol{q}\times\Delta\boldsymbol{v}) + (q_0\Delta\boldsymbol{v}\times\boldsymbol{q} - (\boldsymbol{q}\times\Delta\boldsymbol{v})\times\boldsymbol{q}) \end{Bmatrix} \\ &= \begin{Bmatrix} (\boldsymbol{q}\times\Delta\boldsymbol{v})\cdot\boldsymbol{q} \\ (\boldsymbol{q}\cdot\Delta\boldsymbol{v})\boldsymbol{q} + q_0^2\Delta\boldsymbol{v} - 2q_0\boldsymbol{q}\times\Delta\boldsymbol{v} - (\boldsymbol{q}\times\Delta\boldsymbol{v})\times\boldsymbol{q} \end{Bmatrix} \\ &= \begin{Bmatrix} 0 \\ (q_0^2 - \boldsymbol{q}\cdot\boldsymbol{q})\Delta\boldsymbol{v} - 2q_0\boldsymbol{q}\times\Delta\boldsymbol{v} + 2(\boldsymbol{q}\cdot\Delta\boldsymbol{v})\boldsymbol{q} \end{Bmatrix} \end{aligned}$$

$$\tilde{q} \begin{Bmatrix} 0 \\ \Delta\boldsymbol{v} \end{Bmatrix} \tilde{q}^* = \begin{Bmatrix} 0 \\ (q_0^2 - \boldsymbol{q}\cdot\boldsymbol{q})\Delta\boldsymbol{v} + 2q_0\boldsymbol{q}\times\Delta\boldsymbol{v} + 2(\boldsymbol{q}\cdot\Delta\boldsymbol{v})\boldsymbol{q} \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ \Delta\boldsymbol{v}_1 \end{Bmatrix} \begin{Bmatrix} 0 \\ \boldsymbol{v}_2 \end{Bmatrix} = \begin{Bmatrix} -\boldsymbol{v}_2\cdot\Delta\boldsymbol{v}_1 \\ -\boldsymbol{v}_2\times\Delta\boldsymbol{v}_1 \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ \boldsymbol{v}_1 \end{Bmatrix} \begin{Bmatrix} 0 \\ \Delta\boldsymbol{v}_2 \end{Bmatrix} = \begin{Bmatrix} -\boldsymbol{v}_1\cdot\Delta\boldsymbol{v}_2 \\ \boldsymbol{v}_1\times\Delta\boldsymbol{v}_2 \end{Bmatrix}$$

- 速度の運動方程式の線形化

$$
\begin{aligned}
\frac{d}{dt}\begin{Bmatrix}0\\ \Delta\dot{\vec{r}}_e^n\end{Bmatrix} &= \frac{d}{dt}\left(\begin{Bmatrix}0\\ \dot{\vec{r}}_e^n+\Delta\dot{\vec{r}}_e^n\end{Bmatrix}-\begin{Bmatrix}0\\ \dot{\vec{r}}_e^n\end{Bmatrix}\right)\\
&= \Bigg[\left(\tilde{q}_n^b+\Delta\tilde{q}_n^b\right)\begin{Bmatrix}0\\ \vec{a}^b+\Delta\vec{a}^b\end{Bmatrix}\left(\tilde{q}_n^b+\Delta\tilde{q}_n^b\right)^*+\begin{Bmatrix}0\\ \vec{g}^n+\Delta\vec{g}^n\end{Bmatrix}\\
&\qquad\qquad -\begin{Bmatrix}0\\ \left(2\left(\vec{\omega}_{e/i}^n+\Delta\vec{\omega}_{e/i}^n\right)+\left(\vec{\omega}_{n/e}^n+\Delta\vec{\omega}_{n/e}^n\right)\right)\times\left(\dot{\vec{r}}_e^n+\Delta\dot{\vec{r}}_e^n\right)\end{Bmatrix}\\
&\qquad\qquad -\left(\tilde{q}_e^n+\Delta\tilde{q}_e^n\right)^*\begin{Bmatrix}0\\ \vec{\omega}_{e/i}^e\times\left(\vec{\omega}_{e/i}^e\times\left(\vec{r}_e+\Delta\vec{r}_e\right)\right)\end{Bmatrix}\left(\tilde{q}_e^n+\Delta\tilde{q}_e^n\right)\Bigg]\\
&\qquad -\Bigg[\tilde{q}_n^b\begin{Bmatrix}0\\ \vec{a}^b\end{Bmatrix}\tilde{q}_n^{b*}+\begin{Bmatrix}0\\ \vec{g}^n\end{Bmatrix}\\
&\qquad\qquad -\begin{Bmatrix}0\\ \left(2\vec{\omega}_{e/i}^n+\vec{\omega}_{n/e}^n\right)\times\dot{\vec{r}}_e^n\end{Bmatrix}\\
&\qquad\qquad -\tilde{q}_e^{n*}\begin{Bmatrix}0\\ \vec{\omega}_{e/i}^e\times\left(\vec{\omega}_{e/i}^e\times\vec{r}_e\right)\end{Bmatrix}\tilde{q}_e^n\Bigg]\\
&= \Delta\tilde{q}_n^b\begin{Bmatrix}0\\ \vec{a}^b\end{Bmatrix}\tilde{q}_n^{b*}+\tilde{q}_n^b\begin{Bmatrix}0\\ \vec{a}^b\end{Bmatrix}\Delta\tilde{q}_n^{b*}+\tilde{q}_n^b\begin{Bmatrix}0\\ \Delta\vec{a}^b\end{Bmatrix}\tilde{q}_n^{b*}+\begin{Bmatrix}0\\ \Delta\vec{g}^n\end{Bmatrix}\\
&\qquad -\begin{Bmatrix}0\\ \left(2\Delta\vec{\omega}_{e/i}^n+\Delta\vec{\omega}_{n/e}^n\right)\times\dot{\vec{r}}_e^n+\left(2\vec{\omega}_{e/i}^n+\vec{\omega}_{n/e}^n\right)\times\Delta\dot{\vec{r}}_e^n\end{Bmatrix}\\
&\qquad -\tilde{q}_e^{n*}\begin{Bmatrix}0\\ \vec{\omega}_{e/i}^e\times\left(\vec{\omega}_{e/i}^e\times\Delta\vec{r}_e\right)\end{Bmatrix}\tilde{q}_e^n\\
&\qquad -\Delta\tilde{q}_e^{n*}\begin{Bmatrix}0\\ \vec{\omega}_{e/i}^e\times\left(\vec{\omega}_{e/i}^e\times\vec{r}_e\right)\end{Bmatrix}\tilde{q}_e^n-\tilde{q}_e^{n*}\begin{Bmatrix}0\\ \vec{\omega}_{e/i}^e\times\left(\vec{\omega}_{e/i}^e\times\vec{r}_e\right)\end{Bmatrix}\Delta\tilde{q}_e^n
\end{aligned}
$$

* $\Delta\vec{\omega}_{e/i}^n$ は

$$
\begin{aligned}
\begin{Bmatrix}0\\ \Delta\vec{\omega}_{e/i}^n\end{Bmatrix} &= \left(\tilde{q}_e^n+\Delta\tilde{q}_e^n\right)^*\begin{Bmatrix}0\\ \vec{\omega}_{e/i}^i\end{Bmatrix}\left(\tilde{q}_e^n+\Delta\tilde{q}_e^n\right)-\tilde{q}_e^{n*}\begin{Bmatrix}0\\ \vec{\omega}_{e/i}^i\end{Bmatrix}\tilde{q}_e^n\\
&= \Delta\tilde{q}_e^{n*}\begin{Bmatrix}0\\ \vec{\omega}_{e/i}^i\end{Bmatrix}\tilde{q}_e^n+\tilde{q}_e^{n*}\begin{Bmatrix}0\\ \vec{\omega}_{e/i}^i\end{Bmatrix}\Delta\tilde{q}_e^n\\
&= \Delta\tilde{q}_e^{n*}\begin{Bmatrix}0\\ \begin{pmatrix}0\\0\\ \Omega_{e/i}\end{pmatrix}\end{Bmatrix}\tilde{q}_e^n+\tilde{q}_e^{n*}\begin{Bmatrix}0\\ \begin{pmatrix}0\\0\\ \Omega_{e/i}\end{pmatrix}\end{Bmatrix}\Delta\tilde{q}_e^n\\
&= 2\begin{Bmatrix}0\\ \left(q_0\begin{pmatrix}0\\0\\ \Omega_{e/i}\end{pmatrix}+\begin{pmatrix}0\\0\\ \Omega_{e/i}\end{pmatrix}\boldsymbol{q}\right)\Delta q_0+\left(\boldsymbol{q}\cdot\begin{pmatrix}0\\0\\ \Omega_{e/i}\end{pmatrix}\right)\Delta\boldsymbol{q}+\left(q_0\begin{pmatrix}0\\0\\ \Omega_{e/i}\end{pmatrix}+\begin{pmatrix}0\\0\\ \Omega_{e/i}\end{pmatrix}\times\boldsymbol{q}\right)\times\Delta\boldsymbol{q}\end{Bmatrix}\\
&= 2\Omega_{e/i}\begin{Bmatrix}0\\ -q_2\Delta q_0+q_3\Delta q_1-q_0\Delta q_2+q_1\Delta q_3\\ q_1\Delta q_0+q_0\Delta q_1+q_3\Delta q_2+q_2\Delta q_3\\ q_0\Delta q_0-q_1\Delta q_1-q_2\Delta q_2+q_3\Delta q_3\end{Bmatrix}
\end{aligned}
$$

* $\Delta\vec{\omega}^n_{n/e}$ は

$$\Delta\vec{\omega}^n_{n/e} = \frac{1}{r_e + (h+\Delta h)} \begin{pmatrix} (\dot{r}^n_e)_Y + \Delta(\dot{r}^n_e)_Y \\ -(\dot{r}^n_e)_X - \Delta(\dot{r}^n_e)_X \\ 0 \end{pmatrix} - \frac{1}{r_e+h} \begin{pmatrix} (\dot{r}^n_e)_Y \\ -(\dot{r}^n_e)_X \\ 0 \end{pmatrix}$$

$$= \frac{1}{r_e+h} \begin{pmatrix} \Delta(\dot{r}^n_e)_Y \\ -\Delta(\dot{r}^n_e)_X \\ 0 \end{pmatrix} - \frac{1}{(r_e+h)^2} \begin{pmatrix} (\dot{r}^n_e)_Y \\ -(\dot{r}^n_e)_X \\ 0 \end{pmatrix} \Delta h$$

* $\vec{\omega}^e_{e/i} \times \left(\vec{\omega}^e_{e/i} \times \vec{r}_e\right)$ に関する部分は

$$\Delta\left[\tilde{q}^{n*}_e \begin{Bmatrix} 0 \\ \vec{\omega}^e_{e/i} \times \left(\vec{\omega}^e_{e/i} \times \vec{r}_e\right) \end{Bmatrix} \tilde{q}^n_e\right] = \tilde{q}^{n*}_e \begin{Bmatrix} 0 \\ \vec{\omega}^e_{e/i} \times \left(\vec{\omega}^e_{e/i} \times \Delta\vec{r}_e\right) \end{Bmatrix} \tilde{q}^n_e$$

$$+ \Delta\tilde{q}^{n*}_e \begin{Bmatrix} 0 \\ \vec{\omega}^e_{e/i} \times \left(\vec{\omega}^e_{e/i} \times \vec{r}_e\right) \end{Bmatrix} \tilde{q}^n_e + \tilde{q}^{n*}_e \begin{Bmatrix} 0 \\ \vec{\omega}^e_{e/i} \times \left(\vec{\omega}^e_{e/i} \times \vec{r}_e\right) \end{Bmatrix} \Delta\tilde{q}^n_e$$

ここで

$$\tilde{q}^{n*}_e \begin{Bmatrix} 0 \\ \vec{\omega}^e_{e/i} \times \left(\vec{\omega}^e_{e/i} \times \Delta\vec{r}_e\right) \end{Bmatrix} \tilde{q}^n_e$$

$$= 2{\Omega_{e/i}}^2 \mathrm{Rot}\left[\tilde{q}^n_e\right] \left\{ \begin{pmatrix} q_0q_2 + q_1q_3 \\ q_3q_2 - q_1q_0 \\ 0 \end{pmatrix}_{\tilde{q}^n_e} \Delta h + (R_{\mathrm{normal}} + h) \begin{bmatrix} q_2 & q_3 & q_0 & q_1 \\ -q_1 & -q_0 & q_3 & q_2 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{\tilde{q}^n_e} \Delta\tilde{q}^n_e \right\}$$

$$\Delta\tilde{q}^{n*}_e \begin{Bmatrix} 0 \\ \vec{\omega}^e_{e/i} \times \left(\vec{\omega}^e_{e/i} \times \vec{r}_e\right) \end{Bmatrix} \tilde{q}^n_e + \tilde{q}^{n*}_e \begin{Bmatrix} 0 \\ \vec{\omega}^e_{e/i} \times \left(\vec{\omega}^e_{e/i} \times \vec{r}_e\right) \end{Bmatrix} \Delta\tilde{q}^n_e$$

$$= 2 \left\{ \begin{matrix} \begin{bmatrix} \omega_1 & \lambda & -\omega_3 & \omega_2 \\ \omega_2 & \omega_3 & \lambda & -\omega_1 \\ \omega_3 & -\omega_2 & \omega_1 & \lambda \end{bmatrix} & \begin{matrix} 0 \\ \\ \lambda = \boldsymbol{q}^n_e \cdot \boldsymbol{v} \\ \omega = q^n_{e0}\boldsymbol{v} + \boldsymbol{v} \times \boldsymbol{q}^n_e \\ \boldsymbol{v} = \vec{\omega}^e_{e/i} \times \left(\vec{\omega}^e_{e/i} \times \vec{r}_e\right) \end{matrix} & \Delta\tilde{q}^n_e \end{matrix} \right\}$$

$$\lambda = 2{\Omega_{e/i}}^2 (R_{\mathrm{normal}} + h) \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}_{\boldsymbol{q}^n_e} \cdot \begin{pmatrix} q_0q_2 + q_1q_3 \\ q_3q_2 - q_1q_0 \\ 0 \end{pmatrix}_{\tilde{q}^n_e}$$

$$= 2{\Omega_{e/i}}^2 (R_{\mathrm{normal}} + h)\, q_3 \left({q_1}^2 + {q_2}^2\right)$$

$$\omega = 2{\Omega_{e/i}}^2 (R_{\mathrm{normal}} + h) \left(q_0 \begin{pmatrix} q_0q_2 + q_1q_3 \\ q_3q_2 - q_1q_0 \\ 0 \end{pmatrix}_{\tilde{q}^n_e} + \begin{pmatrix} q_0q_2 + q_1q_3 \\ q_3q_2 - q_1q_0 \\ 0 \end{pmatrix}_{\tilde{q}^n_e} \times \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}_{\tilde{q}^n_e} \right)$$

$$= 2{\Omega_{e/i}}^2 (R_{\mathrm{normal}} + h) \begin{pmatrix} q_2\left({q_0}^2 + {q_3}^2\right) \\ -q_1\left({q_0}^2 + {q_3}^2\right) \\ q_0\left({q_1}^2 + {q_2}^2\right) \end{pmatrix}_{\tilde{q}^n_e}$$

- 位置の運動方程式の線形化

$$\frac{d}{dt}\Delta\tilde{q}^n_e = \frac{1}{2}\left(\tilde{q}^n_e + \Delta\tilde{q}^n_e\right) \begin{Bmatrix} 0 \\ \vec{\omega}^n_{n/e} + \Delta\vec{\omega}^n_{n/e} \end{Bmatrix} - \frac{1}{2}\tilde{q}^n_e \begin{Bmatrix} 0 \\ \vec{\omega}^n_{n/e} \end{Bmatrix}$$

$$= \frac{1}{2}\tilde{q}^n_e \begin{Bmatrix} 0 \\ \Delta\vec{\omega}^n_{n/e} \end{Bmatrix} + \frac{1}{2}\Delta\tilde{q}^n_e \begin{Bmatrix} 0 \\ \vec{\omega}^n_{n/e} \end{Bmatrix}$$

$$
\begin{aligned}
\frac{d}{dt}\Delta h &= -((\dot{r}_e^n)_Z + \Delta(\dot{r}_e^n)_Z) - (-(\dot{r}_e^n)_Z) \\
&= -\Delta(\dot{r}_e^n)_Z
\end{aligned}
$$

- 姿勢の運動方程式の線形化

$$
\begin{aligned}
\frac{d}{dt}\tilde{q}_n^b = \frac{1}{2}\Bigg[&\left(\tilde{q}_n^b + \Delta\tilde{q}_n^b\right)\begin{Bmatrix} 0 \\ \vec{\omega}_{b/i}^b + \Delta\vec{\omega}_{b/i}^b \end{Bmatrix} \\
&- \left(\begin{Bmatrix} 0 \\ \vec{\omega}_{e/i}^n + \Delta\vec{\omega}_{e/i}^n \end{Bmatrix} + \begin{Bmatrix} 0 \\ \vec{\omega}_{n/e}^n + \Delta\vec{\omega}_{n/e}^n \end{Bmatrix}\right)\left(\tilde{q}_n^b + \Delta\tilde{q}_n^b\right)\Bigg] \\
&- \frac{1}{2}\left[\tilde{q}_n^b \begin{Bmatrix} 0 \\ \vec{\omega}_{b/i}^b \end{Bmatrix} - \left(\begin{Bmatrix} 0 \\ \vec{\omega}_{e/i}^n \end{Bmatrix} + \begin{Bmatrix} 0 \\ \vec{\omega}_{n/e}^n \end{Bmatrix}\right)\tilde{q}_n^b\right] \\
= \frac{1}{2}\Bigg[&\Delta\tilde{q}_n^b \begin{Bmatrix} 0 \\ \vec{\omega}_{b/i}^b \end{Bmatrix} + \tilde{q}_n^b \begin{Bmatrix} 0 \\ \Delta\vec{\omega}_{b/i}^b \end{Bmatrix} \\
&- \left(\begin{Bmatrix} 0 \\ \vec{\omega}_{e/i}^n \end{Bmatrix} + \begin{Bmatrix} 0 \\ \vec{\omega}_{n/e}^n \end{Bmatrix}\right)\Delta\tilde{q}_n^b - \left(\begin{Bmatrix} 0 \\ \Delta\vec{\omega}_{e/i}^n \end{Bmatrix} + \begin{Bmatrix} 0 \\ \Delta\vec{\omega}_{n/e}^n \end{Bmatrix}\right)\tilde{q}_n^b\Bigg]
\end{aligned}
$$

2 Quarternion Error Model

特殊な演算を考えることにする

$$
\exp(\boldsymbol{r}) = \begin{cases} \begin{Bmatrix} \cos(|\boldsymbol{r}|) \\ \sin(|\boldsymbol{r}|)\frac{\boldsymbol{r}}{|\boldsymbol{r}|} \end{Bmatrix} & (\boldsymbol{r} \neq \boldsymbol{0}) \\ \begin{Bmatrix} 1 \\ \boldsymbol{0} \end{Bmatrix} & (\boldsymbol{r} = \boldsymbol{0}) \end{cases} \tag{1}
$$

$$
\tilde{q} + \Delta\tilde{q} \equiv \exp(\boldsymbol{u}) * \tilde{q} \tag{2}
$$

$$
\begin{aligned}
(\tilde{q} + \Delta\tilde{q})^* &\equiv (\exp(\boldsymbol{u}) * \tilde{q})^* \\
&= \tilde{q}^* * \exp(\boldsymbol{u})^* \\
&= \tilde{q}^* * \exp(-\boldsymbol{u})
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
(\tilde{q} + \Delta\tilde{q})^* \begin{Bmatrix} 0 \\ \boldsymbol{v} \end{Bmatrix} (\tilde{q} + \Delta\tilde{q}) &\equiv (\exp(\boldsymbol{u}) * \tilde{q})^* \begin{Bmatrix} 0 \\ \boldsymbol{v} \end{Bmatrix} (\exp(\boldsymbol{u}) * \tilde{q}) \\
&= \tilde{q}^* \exp(-\boldsymbol{u}) \begin{Bmatrix} 0 \\ \boldsymbol{v} \end{Bmatrix} \exp(\boldsymbol{u})\tilde{q}
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
\tilde{q}^* \begin{Bmatrix} 0 \\ \boldsymbol{v} \end{Bmatrix} \tilde{q} &= \begin{Bmatrix} q_0 \\ -\boldsymbol{q} \end{Bmatrix} \begin{Bmatrix} 0 \\ \boldsymbol{v} \end{Bmatrix} \begin{Bmatrix} q_0 \\ \boldsymbol{q} \end{Bmatrix} \\
&= \begin{Bmatrix} 0 \\ (q_0^2 - \boldsymbol{q}\cdot\boldsymbol{q})\boldsymbol{v} - 2q_0\boldsymbol{q}\times\boldsymbol{v} + 2(\boldsymbol{q}\cdot\boldsymbol{v})\boldsymbol{q} \end{Bmatrix}
\end{aligned}
$$

より

$$
\begin{aligned}
\exp(-\boldsymbol{u})\begin{Bmatrix}0\\ \boldsymbol{v}\end{Bmatrix}\exp(\boldsymbol{u}) &= \begin{Bmatrix}\cos(|-\boldsymbol{u}|)\\ \sin(|-\boldsymbol{u}|)\frac{-\boldsymbol{u}}{|-\boldsymbol{u}|}\end{Bmatrix}\begin{Bmatrix}0\\ \boldsymbol{v}\end{Bmatrix}\begin{Bmatrix}\cos(|\boldsymbol{u}|)\\ \sin(|\boldsymbol{u}|)\frac{\boldsymbol{u}}{|\boldsymbol{u}|}\end{Bmatrix}\\
&= \begin{Bmatrix}\cos(|\boldsymbol{u}|)\\ \sin(|\boldsymbol{u}|)\frac{-\boldsymbol{u}}{|\boldsymbol{u}|}\end{Bmatrix}\begin{Bmatrix}0\\ \boldsymbol{v}\end{Bmatrix}\begin{Bmatrix}\cos(|\boldsymbol{u}|)\\ \sin(|\boldsymbol{u}|)\frac{\boldsymbol{u}}{|\boldsymbol{u}|}\end{Bmatrix}\\
&= \begin{Bmatrix}0\\ (\cos(|\boldsymbol{u}|)^2-\sin(|\boldsymbol{u}|)^2\frac{-\boldsymbol{u}\cdot\boldsymbol{u}}{|\boldsymbol{u}|^2})\boldsymbol{v}-2\cos(|\boldsymbol{u}|)\sin(|\boldsymbol{u}|)\frac{\boldsymbol{u}}{|\boldsymbol{u}|}\times\boldsymbol{v}+2(\sin(|\boldsymbol{u}|)\frac{\boldsymbol{u}}{|\boldsymbol{u}|}\cdot\boldsymbol{v})\sin(|\boldsymbol{u}|)\frac{\boldsymbol{u}}{|\boldsymbol{u}|}\end{Bmatrix}\\
&= \begin{Bmatrix}0\\ \boldsymbol{v}-2\cos(|\boldsymbol{u}|)\sin(|\boldsymbol{u}|)\frac{\boldsymbol{u}}{|\boldsymbol{u}|}\times\boldsymbol{v}+2(\sin(|\boldsymbol{u}|)\frac{\boldsymbol{u}}{|\boldsymbol{u}|}\cdot\boldsymbol{v})\sin(|\boldsymbol{u}|)\frac{\boldsymbol{u}}{|\boldsymbol{u}|}\end{Bmatrix}
\end{aligned}
\tag{5}
$$

ここで $|\boldsymbol{u}| \ll 1$ の仮定を用いると

$$
\frac{\sin(|\boldsymbol{u}|)}{|\boldsymbol{u}|} \approx 1 \tag{6}
$$

$$
\frac{\sin(|\boldsymbol{u}|)\cos(|\boldsymbol{u}|)}{|\boldsymbol{u}|} \approx 1 \tag{7}
$$

$$
(\boldsymbol{u}\cdot\boldsymbol{v})\boldsymbol{u} \approx \boldsymbol{0} \tag{8}
$$

であるから、

$$
\begin{aligned}
\exp(-\boldsymbol{u})\begin{Bmatrix}0\\ \boldsymbol{v}\end{Bmatrix}\exp(\boldsymbol{u}) &= \begin{Bmatrix}0\\ \boldsymbol{v}-2\cos(|\boldsymbol{u}|)\sin(|\boldsymbol{u}|)\frac{\boldsymbol{u}}{|\boldsymbol{u}|}\times\boldsymbol{v}+2(\sin(|\boldsymbol{u}|)\frac{\boldsymbol{u}}{|\boldsymbol{u}|}\cdot\boldsymbol{v})\sin(|\boldsymbol{u}|)\frac{\boldsymbol{u}}{|\boldsymbol{u}|}\end{Bmatrix}\\
&\approx \begin{Bmatrix}0\\ \boldsymbol{v}+2\boldsymbol{v}\times\boldsymbol{u}\end{Bmatrix}\\
&= \begin{Bmatrix}0\\ \boldsymbol{v}+2\begin{bmatrix}0 & -v_2 & v_1\\ v_2 & 0 & -v_0\\ -v_1 & v_0 & 0\end{bmatrix}_{\boldsymbol{v}}\boldsymbol{u}\end{Bmatrix}
\end{aligned}
\tag{9}
$$

2.1　速度の運動方程式の線形化

$$
\begin{aligned}
\frac{d}{dt}\begin{Bmatrix} 0 \\ \Delta\dot{\vec{r}}_e^n \end{Bmatrix} &= \frac{d}{dt}\left(\begin{Bmatrix} 0 \\ \dot{\vec{r}}_e^n + \Delta\dot{\vec{r}}_e^n \end{Bmatrix} - \begin{Bmatrix} 0 \\ \dot{\vec{r}}_e^n \end{Bmatrix}\right) \\
&= \Bigg[(\tilde{q}_b^n + \Delta\tilde{q}_b^n)^* \begin{Bmatrix} 0 \\ \vec{a}^b + \Delta\vec{a}^b \end{Bmatrix} (\tilde{q}_b^n + \Delta\tilde{q}_b^n) + \begin{Bmatrix} 0 \\ \vec{g}^n + \Delta\vec{g}^n \end{Bmatrix} \\
&\qquad - \begin{Bmatrix} 0 \\ \left(2\left(\vec{\omega}_{e/i}^n + \Delta\vec{\omega}_{e/i}^n\right) + \left(\vec{\omega}_{n/e}^n + \Delta\vec{\omega}_{n/e}^n\right)\right) \times \left(\dot{\vec{r}}_e^n + \Delta\dot{\vec{r}}_e^n\right) \end{Bmatrix} \\
&\qquad - (\tilde{q}_e^n + \Delta\tilde{q}_e^n)^* \begin{Bmatrix} 0 \\ \vec{\omega}_{e/i}^e \times \left(\vec{\omega}_{e/i}^e \times (\vec{r}_e + \Delta\vec{r}_e)\right) \end{Bmatrix} (\tilde{q}_e^n + \Delta\tilde{q}_e^n)\Bigg] \\
&\quad - \left[\tilde{q}_b^{n*} \begin{Bmatrix} 0 \\ \vec{a}^b \end{Bmatrix} \tilde{q}_b^n + \begin{Bmatrix} 0 \\ \vec{g}^n \end{Bmatrix} - \begin{Bmatrix} 0 \\ \left(2\vec{\omega}_{e/i}^n + \vec{\omega}_{n/e}^n\right) \times \dot{\vec{r}}_e^n \end{Bmatrix} - \tilde{q}_e^{n*} \begin{Bmatrix} 0 \\ \vec{\omega}_{e/i}^e \times \left(\vec{\omega}_{e/i}^e \times \vec{r}_e\right) \end{Bmatrix} \tilde{q}_e^n\right] \\
&= \left(\exp(\boldsymbol{u}_n^b) * \tilde{q}_n^b\right) \begin{Bmatrix} 0 \\ \vec{a}^b + \Delta\vec{a}^b \end{Bmatrix} \left(\exp(\boldsymbol{u}_n^b) * \tilde{q}_n^b\right)^* - \tilde{q}_n^b \begin{Bmatrix} 0 \\ \vec{a}^b \end{Bmatrix} \tilde{q}_n^{b*} + \begin{Bmatrix} 0 \\ \Delta\vec{g}^n \end{Bmatrix} \\
&\quad - \begin{Bmatrix} 0 \\ \left(2\Delta\vec{\omega}_{e/i}^n + \Delta\vec{\omega}_{n/e}^n\right) \times \dot{\vec{r}}_e^n + \left(2\vec{\omega}_{e/i}^n + \vec{\omega}_{n/e}^n\right) \times \Delta\dot{\vec{r}}_e^n \end{Bmatrix} \\
&\quad - (\exp(\boldsymbol{u}_e^n) * \tilde{q}_e^n)^* \begin{Bmatrix} 0 \\ \vec{\omega}_{e/i}^e \times \left(\vec{\omega}_{e/i}^e \times \vec{r}_e\right) \end{Bmatrix} (\exp(\boldsymbol{u}_e^n) * \tilde{q}_e^n) + \tilde{q}_e^{n*} \begin{Bmatrix} 0 \\ \vec{\omega}_{e/i}^e \times \left(\vec{\omega}_{e/i}^e \times \vec{r}_e\right) \end{Bmatrix} \tilde{q}_e^n \\
&\quad \tilde{q}_e^{n*} \begin{Bmatrix} 0 \\ \vec{\omega}_{e/i}^e \times \left(\vec{\omega}_{e/i}^e \times \Delta\vec{r}_e\right) \end{Bmatrix} \tilde{q}_e^n \\
&= \exp(\boldsymbol{u}_n^b) \begin{Bmatrix} 0 \\ \mathrm{Rot}\left[\tilde{q}_n^{b*}\right] \left(\vec{a}^b + \Delta\vec{a}^b\right) \end{Bmatrix} \exp(-\boldsymbol{u}_n^b) - \begin{Bmatrix} 0 \\ \mathrm{Rot}\left[\tilde{q}_n^{b*}\right] \vec{a}^b \end{Bmatrix} + \begin{Bmatrix} 0 \\ \Delta\vec{g}^n \end{Bmatrix} \\
&\quad - \begin{Bmatrix} 0 \\ \left(2\Delta\vec{\omega}_{e/i}^n + \Delta\vec{\omega}_{n/e}^n\right) \times \dot{\vec{r}}_e^n + \left(2\vec{\omega}_{e/i}^n + \vec{\omega}_{n/e}^n\right) \times \Delta\dot{\vec{r}}_e^n \end{Bmatrix} \\
&\quad - \tilde{q}_e^{n*} \left\{ -\exp\left(\boldsymbol{u}_e^n\right) \begin{Bmatrix} 0 \\ \vec{\omega}_{e/i}^e \times \left(\vec{\omega}_{e/i}^e \times \vec{r}_e\right) \end{Bmatrix} \exp\left(\boldsymbol{u}_e^n\right) - \begin{Bmatrix} 0 \\ \vec{\omega}_{e/i}^e \times \left(\vec{\omega}_{e/i}^e \times \vec{r}_e\right) \end{Bmatrix} \right\} \tilde{q}_e^n \\
&\quad - \begin{Bmatrix} 0 \\ \mathrm{Rot}\left[\tilde{q}_e^n\right] \left(\vec{\omega}_{e/i}^e \times \left(\vec{\omega}_{e/i}^e \times \Delta\vec{r}_e\right)\right) \end{Bmatrix} \\
&\approx \begin{Bmatrix} 0 \\ \mathrm{Rot}\left[\tilde{q}_n^{b*}\right] \Delta\vec{a}^b - 2\mathrm{Rot}\left[\tilde{q}_n^{b*}\right] \vec{a}^b \times \boldsymbol{u}_n^b \end{Bmatrix} + \begin{Bmatrix} 0 \\ \Delta\vec{g}^n \end{Bmatrix} \\
&\quad - \begin{Bmatrix} 0 \\ \left(2\Delta\vec{\omega}_{e/i}^n + \Delta\vec{\omega}_{n/e}^n\right) \times \dot{\vec{r}}_e^n + \left(2\vec{\omega}_{e/i}^n + \vec{\omega}_{n/e}^n\right) \times \Delta\dot{\vec{r}}_e^n \end{Bmatrix} \\
&\quad - \begin{Bmatrix} 0 \\ 2\mathrm{Rot}\left[\tilde{q}_e^n\right] \left(\left(\vec{\omega}_{e/i}^e \times \left(\vec{\omega}_{e/i}^e \times \vec{r}_e\right)\right) \times \boldsymbol{u}_e^n\right) \end{Bmatrix} - \begin{Bmatrix} 0 \\ \mathrm{Rot}\left[\tilde{q}_e^n\right] \left(\vec{\omega}_{e/i}^e \times \left(\vec{\omega}_{e/i}^e \times \Delta\vec{r}_e\right)\right) \end{Bmatrix}
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
\begin{Bmatrix} 0 \\ \Delta\vec{\omega}^n_{e/i} \end{Bmatrix} &= (\tilde{q}^n_e + \Delta\tilde{q}^n_e)^* \begin{Bmatrix} 0 \\ \vec{\omega}^i_{e/i} \end{Bmatrix} (\tilde{q}^n_e + \Delta\tilde{q}^n_e) - \tilde{q}^{n*}_e \begin{Bmatrix} 0 \\ \vec{\omega}^i_{e/i} \end{Bmatrix} \tilde{q}^n_e \\
&\equiv (\exp(\boldsymbol{u}^n_e) * \tilde{q}^n_e)^* \begin{Bmatrix} 0 \\ \vec{\omega}^i_{e/i} \end{Bmatrix} (\exp(\boldsymbol{u}^n_e) * \tilde{q}^n_e) - \tilde{q}^{n*}_e \begin{Bmatrix} 0 \\ \vec{\omega}^i_{e/i} \end{Bmatrix} \tilde{q}^n_e \\
&= \tilde{q}^{n*}_e \exp(-\boldsymbol{u}^n_e) \begin{Bmatrix} 0 \\ \vec{\omega}^i_{e/i} \end{Bmatrix} \exp(\boldsymbol{u}^n_e) \tilde{q}^n_e - \tilde{q}^{n*}_e \begin{Bmatrix} 0 \\ \vec{\omega}^i_{e/i} \end{Bmatrix} \tilde{q}^n_e
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
\begin{Bmatrix} 0 \\ \Delta\vec{\omega}^n_{e/i} \end{Bmatrix} &\approx \tilde{q}^{n*}_e \begin{Bmatrix} 0 \\ \vec{\omega}^i_{e/i} + 2\vec{\omega}^i_{e/i} \times \boldsymbol{u}^n_e \end{Bmatrix} \tilde{q}^n_e - \tilde{q}^{n*}_e \begin{Bmatrix} 0 \\ \vec{\omega}^i_{e/i} \end{Bmatrix} \tilde{q}^n_e \\
&= \tilde{q}^{n*}_e \begin{Bmatrix} 0 \\ 2\vec{\omega}^i_{e/i} \times \boldsymbol{u}^n_e \end{Bmatrix} \tilde{q}^n_e
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
\Delta\vec{\omega}^n_{e/i} &= \mathrm{Rot}[\tilde{q}^n_e]\,(2\vec{\omega}^i_{e/i} \times \boldsymbol{u}^n_e) \\
&= \begin{bmatrix} R_{00} & R_{01} & R_{02} \\ R_{10} & R_{11} & R_{12} \\ R_{20} & R_{21} & R_{22} \end{bmatrix}_{\tilde{q}^n_e} 2\Omega_{e/i} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{u}^n_e \\
&= 2\Omega_{e/i} \begin{bmatrix} R_{01} & -R_{00} & 0 \\ R_{11} & -R_{10} & 0 \\ R_{21} & -R_{20} & 0 \end{bmatrix}_{\tilde{q}^n_e} \boldsymbol{u}^n_e
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
\Delta\vec{\omega}^n_{n/e} &= \begin{pmatrix} \frac{\Delta(\dot{r}^n_e)_Y}{R_{\mathrm{normal}}+h} \\ -\frac{\Delta(\dot{r}^n_e)_X}{R_{\mathrm{meridian}}+h} \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{(\dot{r}^n_e)_Y}{(R_{\mathrm{normal}}+h)^2} \\ -\frac{(\dot{r}^n_e)_X}{(R_{\mathrm{meridian}}+h)^2} \\ 0 \end{pmatrix} \Delta h \\
&\approx \frac{1}{r_e+h} \begin{pmatrix} \Delta(\dot{r}^n_e)_Y \\ -\Delta(\dot{r}^n_e)_X \\ 0 \end{pmatrix} - \frac{1}{(r_e+h)^2} \begin{pmatrix} (\dot{r}^n_e)_Y \\ -(\dot{r}^n_e)_X \\ 0 \end{pmatrix} \Delta h \\
&= \frac{1}{r_e+h} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Delta\dot{\vec{r}}^n_e - \frac{1}{(r_e+h)^2} \begin{pmatrix} (\dot{r}^n_e)_Y \\ -(\dot{r}^n_e)_X \\ 0 \end{pmatrix} \Delta h
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
\vec{\omega}^e_{e/i} \times \left(\vec{\omega}^e_{e/i} \times \vec{r}_e\right) &= 2{\Omega_{e/i}}^2 (R_{\mathrm{normal}} + h) \begin{pmatrix} q_0q_2 + q_1q_3 \\ q_3q_2 - q_1q_0 \\ 0 \end{pmatrix} \\
&= {\Omega_{e/i}}^2 (R_{\mathrm{normal}} + h) \begin{bmatrix} R_{20} \\ R_{21} \\ 0 \end{bmatrix}_{\mathrm{Rot}[\tilde{q}^n_e]}
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
2\mathrm{Rot}\left[\tilde{q}_e^n\right]&\left(\left(\vec{\omega}_{e/i}^e \times \left(\vec{\omega}_{e/i}^e \times \vec{r}_e\right)\right) \times \boldsymbol{u}_e^n\right) \\
&= 2\mathrm{Rot}\left[\tilde{q}_e^n\right] \begin{bmatrix} 0 & -v_2 & v_1 \\ v_2 & 0 & -v_0 \\ -v_1 & v_0 & 0 \end{bmatrix}_{\vec{\omega}_{e/i}^e \times \left(\vec{\omega}_{e/i}^e \times \vec{r}_e\right)} \boldsymbol{u}_e^n \\
&= 2{\Omega_{e/i}}^2 \left(R_{\mathrm{normal}} + h\right) \mathrm{Rot}\left[\tilde{q}_e^n\right] \begin{bmatrix} 0 & 0 & R_{21} \\ 0 & 0 & -R_{20} \\ -R_{21} & R_{20} & 0 \end{bmatrix}_{\mathrm{Rot}\left[\tilde{q}_e^n\right]} \boldsymbol{u}_e^n \\
&= 2{\Omega_{e/i}}^2 \left(R_{\mathrm{normal}} + h\right) \begin{bmatrix} -R_{02}R_{21} & R_{02}R_{20} & R_{00}R_{21} - R_{01}R_{20} \\ -R_{12}R_{21} & R_{12}R_{20} & R_{10}R_{21} - R_{11}R_{20} \\ -R_{22}R_{21} & R_{22}R_{20} & R_{20}R_{21} - R_{21}R_{20} \end{bmatrix}_{\mathrm{Rot}\left[\tilde{q}_e^n\right]} \boldsymbol{u}_e^n
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
\Delta\tilde{q} &= \begin{Bmatrix} 1 \\ \boldsymbol{u} \end{Bmatrix} \tilde{q} - \tilde{q} \\
&= \begin{Bmatrix} q_0 - \boldsymbol{u} \cdot \boldsymbol{q} \\ \boldsymbol{q} + q_0\boldsymbol{u} + \boldsymbol{u} \times \boldsymbol{q} \end{Bmatrix} - \begin{Bmatrix} q_0 \\ \boldsymbol{q} \end{Bmatrix} \\
&= \begin{Bmatrix} -\boldsymbol{u} \cdot \boldsymbol{q} \\ q_0\boldsymbol{u} + \boldsymbol{u} \times \boldsymbol{q} \end{Bmatrix} \\
&= \begin{bmatrix} -q_1 & -q_2 & -q_3 \\ q_0 & -q_3 & q_2 \\ q_3 & q_0 & -q_1 \\ -q_2 & q_1 & q_0 \end{bmatrix} \boldsymbol{u}
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
&\mathrm{Rot}\left[\tilde{q}_e^n\right]\left(\vec{\omega}_{e/i}^e \times \left(\vec{\omega}_{e/i}^e \times \Delta\vec{r}_e\right)\right)\\
&\quad = 2{\Omega_{e/i}}^2 \mathrm{Rot}\left[\tilde{q}_e^n\right]\left\{ \begin{pmatrix} q_0q_2+q_1q_3 \\ q_3q_2-q_1q_0 \\ 0 \end{pmatrix}_{\tilde{q}_e^n} \Delta h + \left(R_{\mathrm{normal}}+h\right) \begin{bmatrix} q_2 & q_3 & q_0 & q_1 \\ -q_1 & -q_0 & q_3 & q_2 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{\tilde{q}_e^n} \Delta\tilde{q}_e^n \right\}\\
&\quad = 2{\Omega_{e/i}}^2 \mathrm{Rot}\left[\tilde{q}_e^n\right]\left\{ \begin{pmatrix} q_0q_2+q_1q_3 \\ q_3q_2-q_1q_0 \\ 0 \end{pmatrix}_{\tilde{q}_e^n} \Delta h \right.\\
&\qquad\qquad \left. + \left(R_{\mathrm{normal}}+h\right) \begin{bmatrix} q_2 & q_3 & q_0 & q_1 \\ -q_1 & -q_0 & q_3 & q_2 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{\tilde{q}_e^n} \begin{bmatrix} -q_1 & -q_2 & -q_3 \\ q_0 & -q_3 & q_2 \\ q_3 & q_0 & -q_1 \\ -q_2 & q_1 & q_0 \end{bmatrix}_{\tilde{q}_e^n} \boldsymbol{u} \right\}\\
&\quad = 2{\Omega_{e/i}}^2 \mathrm{Rot}\left[\tilde{q}_e^n\right]\left\{ \begin{pmatrix} q_0q_2+q_1q_3 \\ q_3q_2-q_1q_0 \\ 0 \end{pmatrix}_{\tilde{q}_e^n} \Delta h \right.\\
&\qquad\qquad \left. + \left(R_{\mathrm{normal}}+h\right) \begin{bmatrix} 2\left(q_0q_3-q_1q_2\right) & 1-2\left({q_2}^2+{q_3}^2\right) & 0 \\ 1-2\left({q_0}^2+{q_2}^2\right) & 2\left(q_0q_3+q_1q_2\right) & 0 \\ 0 & 0 & 0 \end{bmatrix}_{\tilde{q}_e^n} \boldsymbol{u} \right\}\\
&\quad = 2{\Omega_{e/i}}^2 \mathrm{Rot}\left[\tilde{q}_e^n\right]\left\{ \frac{1}{2} \begin{bmatrix} R_{20} \\ R_{21} \\ 0 \end{bmatrix}_{\mathrm{Rot}[\tilde{q}_e^n]} \Delta h + \left(R_{\mathrm{normal}}+h\right) \begin{bmatrix} -R_{10} & R_{00} & 0 \\ -R_{11} & R_{01} & 0 \\ 0 & 0 & 0 \end{bmatrix}_{\mathrm{Rot}[\tilde{q}_e^n]} \boldsymbol{u} \right\}\\
&\quad = {\Omega_{e/i}}^2 \left\{ \begin{bmatrix} R_{00}R_{20}-R_{01}R_{21} \\ R_{10}R_{20}-R_{11}R_{21} \\ R_{20}R_{20}-R_{21}R_{21} \end{bmatrix}_{\mathrm{Rot}[\tilde{q}_e^n]} \Delta h \right.\\
&\qquad \left. +2\left(R_{\mathrm{normal}}+h\right) \begin{bmatrix} -R_{00}R_{10}-R_{01}R_{11} & {R_{00}}^2+{R_{01}}^2 & 0 \\ -{R_{10}}^2-{R_{11}}^2 & R_{10}R_{00}+R_{11}R_{01} & 0 \\ -R_{20}R_{10}-R_{21}R_{11} & R_{20}R_{00}+R_{21}R_{01} & 0 \end{bmatrix}_{\mathrm{Rot}[\tilde{q}_e^n]} \boldsymbol{u}_e^n \right\}
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
\frac{d}{dt}\Delta\dot{\vec{r}}_e^n &= \mathrm{Rot}\left[\tilde{q}_n^{b*}\right]\Delta\vec{a}^b - 2\mathrm{Rot}\left[\tilde{q}_n^{b*}\right]\vec{a}^b \times \boldsymbol{u}_n^b + \Delta\vec{g}^n + \dot{\vec{r}}_e^n \times \left(2\Delta\vec{\omega}_{e/i}^n + \Delta\vec{\omega}_{n/e}^n\right) - \left(2\vec{\omega}_{e/i}^n + \vec{\omega}_{n/e}^n\right) \times \Delta\dot{\vec{r}}_e^n \\
&\qquad - 2\mathrm{Rot}\left[\tilde{q}_e^n\right]\left(\left(\vec{\omega}_{e/i}^e \times \left(\vec{\omega}_{e/i}^e \times \vec{r}_e\right)\right) \times \boldsymbol{u}_e^n\right) - \mathrm{Rot}\left[\tilde{q}_e^n\right]\left(\vec{\omega}_{e/i}^e \times \left(\vec{\omega}_{e/i}^e \times \Delta\vec{r}_e\right)\right) \\
&= \mathrm{Rot}\left[\tilde{q}_n^{b*}\right]\Delta\vec{a}^b - 2\begin{pmatrix} R_{00}a_0^b + R_{10}a_1^b + R_{20}a_2^b \\ R_{01}a_0^b + R_{11}a_1^b + R_{21}a_2^b \\ R_{02}a_0^b + R_{12}a_1^b + R_{22}a_2^b \end{pmatrix}_{\tilde{q}_n^b} \times \boldsymbol{u}_n^b + \Delta\vec{g}^n \\
&\quad + \frac{1}{r_e+h}\begin{bmatrix} 0 & -\dot{r}_{nZ}^e & \dot{r}_{nY}^e \\ \dot{r}_{nZ}^e & 0 & -\dot{r}_{nX}^e \\ -\dot{r}_{nY}^e & \dot{r}_{nX}^e & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\Delta\dot{\vec{r}}_e^n \\
&\quad - \frac{1}{(r_e+h)^2}\begin{bmatrix} 0 & -\dot{r}_{nZ}^e & \dot{r}_{nY}^e \\ \dot{r}_{nZ}^e & 0 & -\dot{r}_{nX}^e \\ -\dot{r}_{nY}^e & \dot{r}_{nX}^e & 0 \end{bmatrix}\begin{pmatrix} (\dot{r}_e^n)_Y \\ -(\dot{r}_e^n)_X \\ 0 \end{pmatrix}\Delta h \\
&\quad + 4\Omega_{e/i}\begin{bmatrix} 0 & -\dot{r}_{nZ}^e & \dot{r}_{nY}^e \\ \dot{r}_{nZ}^e & 0 & -\dot{r}_{nX}^e \\ -\dot{r}_{nY}^e & \dot{r}_{nX}^e & 0 \end{bmatrix}\begin{bmatrix} R_{01} & -R_{00} & 0 \\ R_{11} & -R_{10} & 0 \\ R_{21} & -R_{20} & 0 \end{bmatrix}_{\tilde{q}_e^n}\boldsymbol{u}_e^n \\
&\quad - \begin{bmatrix} 0 & -\omega_2 & \omega_1 \\ \omega_2 & 0 & -\omega_0 \\ -\omega_1 & \omega_0 & 0 \end{bmatrix}_{2\vec{\omega}_{e/i}^n + \vec{\omega}_{n/e}^n}\Delta\dot{\vec{r}}_e^n \\
&\quad - 2{\Omega_{e/i}}^2\left(R_{\mathrm{normal}}+h\right)\begin{bmatrix} -R_{02}R_{21} & R_{02}R_{20} & R_{00}R_{21} - R_{01}R_{20} \\ -R_{12}R_{21} & R_{12}R_{20} & R_{10}R_{21} - R_{11}R_{20} \\ -R_{22}R_{21} & R_{22}R_{20} & R_{20}R_{21} - R_{21}R_{20} \end{bmatrix}_{\mathrm{Rot}[\tilde{q}_e^n]}\boldsymbol{u}_e^n \\
&\quad - {\Omega_{e/i}}^2\left\{\begin{bmatrix} R_{00}R_{20} - R_{01}R_{21} \\ R_{10}R_{20} - R_{11}R_{21} \\ R_{20}R_{20} - R_{21}R_{21} \end{bmatrix}_{\mathrm{Rot}[\tilde{q}_e^n]}\Delta h \right. \\
&\qquad\qquad \left. + 2\left(R_{\mathrm{normal}}+h\right)\begin{bmatrix} -R_{00}R_{10} - R_{01}R_{11} & {R_{00}}^2 + {R_{01}}^2 & 0 \\ -{R_{10}}^2 - {R_{11}}^2 & R_{10}R_{00} + R_{11}R_{01} & 0 \\ -R_{20}R_{10} - R_{21}R_{11} & R_{20}R_{00} + R_{21}R_{01} & 0 \end{bmatrix}_{\mathrm{Rot}[\tilde{q}_e^n]}\boldsymbol{u}_e^n\right\} \\
&= \mathrm{Rot}\left[\tilde{q}_n^{b*}\right]\Delta\vec{a}^b - 2\begin{bmatrix} 0 & -R_{02}a_0^b - R_{12}a_1^b - R_{22}a_2^b & R_{01}a_0^b + R_{11}a_1^b + R_{21}a_2^b \\ R_{02}a_0^b + R_{12}a_1^b + R_{22}a_2^b & 0 & -R_{00}a_0^b - R_{10}a_1^b - R_{20}a_2^b \\ -R_{01}a_0^b - R_{11}a_1^b - R_{21}a_2^b & R_{00}a_0^b + R_{10}a_1^b + R_{20}a_2^b & 0 \end{bmatrix}_{\tilde{q}_n^b}\boldsymbol{u}_n^b \\
&\quad + \Delta\vec{g}^n + \frac{1}{r_e+h}\begin{bmatrix} \dot{r}_{nZ}^e & 0 & 0 \\ 0 & \dot{r}_{nZ}^e & 0 \\ -\dot{r}_{nX}^e & -\dot{r}_{nY}^e & 0 \end{bmatrix}\Delta\dot{\vec{r}}_e^n - \frac{1}{(r_e+h)^2}\begin{pmatrix} (\dot{r}_e^n)_X(\dot{r}_e^n)_Z \\ (\dot{r}_e^n)_Y(\dot{r}_e^n)_Z \\ -(\dot{r}_e^n)_X^2 - (\dot{r}_e^n)_Y^2 \end{pmatrix}\Delta h \\
&\quad + 4\Omega_{e/i}\begin{bmatrix} R_{21}(\dot{r}_n^e)_Y - R_{11}(\dot{r}_n^e)_Z & R_{10}(\dot{r}_n^e)_Z - R_{20}(\dot{r}_n^e)_Y & 0 \\ R_{01}(\dot{r}_n^e)_Z - R_{21}(\dot{r}_n^e)_X & R_{20}(\dot{r}_n^e)_X - R_{00}(\dot{r}_n^e)_Z & 0 \\ R_{11}(\dot{r}_n^e)_X - R_{01}(\dot{r}_n^e)_Y & R_{00}(\dot{r}_n^e)_Y - R_{10}(\dot{r}_n^e)_X & 0 \end{bmatrix}_{\tilde{q}_e^n}\boldsymbol{u}_e^n \\
&\quad - \begin{bmatrix} 0 & -\omega_2 & \omega_1 \\ \omega_2 & 0 & -\omega_0 \\ -\omega_1 & \omega_0 & 0 \end{bmatrix}_{2\vec{\omega}_{e/i}^n + \vec{\omega}_{n/e}^n}\Delta\dot{\vec{r}}_e^n - {\Omega_{e/i}}^2\begin{bmatrix} R_{00}R_{20} - R_{01}R_{21} \\ R_{10}R_{20} - R_{11}R_{21} \\ R_{20}R_{20} - R_{21}R_{21} \end{bmatrix}_{\mathrm{Rot}[\tilde{q}_e^n]}\Delta h \\
&\quad + 2{\Omega_{e/i}}^2\left(R_{\mathrm{normal}}+h\right) \\
&\qquad \begin{bmatrix} R_{00}R_{10} + R_{01}R_{11} + R_{02}R_{21} & -R_{02}R_{20} - {R_{00}}^2 - {R_{01}}^2 & -R_{00}R_{21} + R_{01}R_{20} \\ R_{12}R_{21} + {R_{10}}^2 + {R_{11}}^2 & -R_{12}R_{20} - R_{10}R_{00} - R_{11}R_{01} & -R_{10}R_{21} + R_{11}R_{20} \\ R_{22}R_{21} + R_{20}R_{10} + R_{21}R_{11} & -R_{22}R_{20} - R_{20}R_{00} - R_{21}R_{01} & -R_{20}R_{21} + R_{21}R_{20} \end{bmatrix}_{\mathrm{Rot}[\tilde{q}_e^n]}\boldsymbol{u}_e^n
\end{aligned}
\tag{19}
$$

2.2　位置の運動方程式の線形化

$$\begin{aligned}\frac{d}{dt}(\tilde{q}_e^n+\Delta\tilde{q}_e^n) &= \frac{1}{2}(\tilde{q}_e^n+\Delta\tilde{q}_e^n)\begin{Bmatrix}0\\ \vec{\omega}_{n/e}^n+\Delta\vec{\omega}_{n/e}^n\end{Bmatrix}\\ &\equiv \frac{1}{2}\exp(\boldsymbol{u}_e^n)*\tilde{q}_e^n\begin{Bmatrix}0\\ \vec{\omega}_{n/e}^n+\Delta\vec{\omega}_{n/e}^n\end{Bmatrix}\end{aligned} \tag{20}$$

$$\begin{aligned}\frac{d}{dt}(\tilde{q}_e^n+\Delta\tilde{q}_e^n) &\equiv \frac{d}{dt}\left(\exp(\boldsymbol{u}_e^n)*\tilde{q}_e^n\right)\\ &= \frac{d}{dt}\exp(\boldsymbol{u}_e^n)*\tilde{q}_e^n+\exp(\boldsymbol{u}_e^n)*\frac{d}{dt}\tilde{q}_e^n\end{aligned} \tag{21}$$

$$\begin{aligned}\frac{d}{dt}\exp(\boldsymbol{u}_e^n) &= \left(\frac{d}{dt}(\tilde{q}_e^n+\Delta\tilde{q}_e^n)-\exp(\boldsymbol{u}_e^n)*\frac{d}{dt}\tilde{q}_e^n\right)\tilde{q}_e^{n*}\\ &= \left(\frac{1}{2}\exp(\boldsymbol{u}_e^n)*\tilde{q}_e^n\begin{Bmatrix}0\\ \vec{\omega}_{n/e}^n+\Delta\vec{\omega}_{n/e}^n\end{Bmatrix}-\exp(\boldsymbol{u}_e^n)*\frac{1}{2}\tilde{q}_e^n\begin{Bmatrix}0\\ \vec{\omega}_{n/e}^n\end{Bmatrix}\right)\tilde{q}_e^{n*}\\ &= \frac{1}{2}\exp(\boldsymbol{u}_e^n)*\tilde{q}_e^n\begin{Bmatrix}0\\ \Delta\vec{\omega}_{n/e}^n\end{Bmatrix}\tilde{q}_e^{n*}\end{aligned} \tag{22}$$

ここで $|\boldsymbol{u}|\ll 1$ の仮定を用いると

$$\exp(\boldsymbol{u}_e^n)\approx\begin{Bmatrix}1\\ \boldsymbol{u}_e^n\end{Bmatrix} \tag{23}$$

より

$$\frac{d}{dt}\exp(\boldsymbol{u}_e^n)\approx\frac{d}{dt}\begin{Bmatrix}1\\ \boldsymbol{u}_e^n\end{Bmatrix}=\begin{Bmatrix}0\\ \frac{d}{dt}\boldsymbol{u}_e^n\end{Bmatrix} \tag{24}$$

従って

$$\begin{aligned}\frac{d}{dt}\begin{Bmatrix}0\\ \boldsymbol{u}_e^n\end{Bmatrix} &= \frac{1}{2}\begin{Bmatrix}1\\ \boldsymbol{u}_e^n\end{Bmatrix}\tilde{q}_e^n\begin{Bmatrix}0\\ \Delta\vec{\omega}_{n/e}^n\end{Bmatrix}\tilde{q}_e^{n*}\\ &= \frac{1}{2}\begin{Bmatrix}1\\ \boldsymbol{u}_e^n\end{Bmatrix}\begin{Bmatrix}0\\ \mathrm{Rot}\,[\tilde{q}_e^{n*}]\,\Delta\vec{\omega}_{n/e}^n\end{Bmatrix}\\ &= \frac{1}{2}\begin{Bmatrix}0\\ \mathrm{Rot}\,[\tilde{q}_e^{n*}]\,\Delta\vec{\omega}_{n/e}^n\end{Bmatrix}\end{aligned} \tag{25}$$

$$\begin{aligned}\frac{d}{dt}\boldsymbol{u}_e^n &= \frac{1}{2}\mathrm{Rot}\,[\tilde{q}_e^{n*}]\,\Delta\vec{\omega}_{n/e}^n\\ &= \frac{1}{2}\begin{bmatrix}R_{00} & R_{10} & R_{20}\\ R_{01} & R_{11} & R_{21}\\ R_{02} & R_{12} & R_{22}\end{bmatrix}_{\tilde{q}_e^n}\left\{\frac{1}{r_e+h}\begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\Delta\dot{\vec{r}}_e^n-\frac{1}{(r_e+h)^2}\begin{pmatrix}(\dot{r}_e^n)_Y\\ -(\dot{r}_e^n)_X\\ 0\end{pmatrix}\Delta h\right\}\\ &= \frac{1}{2(r_e+h)}\begin{bmatrix}-R_{10} & R_{00} & 0\\ -R_{11} & R_{01} & 0\\ -R_{12} & R_{02} & 0\end{bmatrix}_{\tilde{q}_e^n}\Delta\dot{\vec{r}}_e^n+\frac{1}{2\,(r_e+h)^2}\begin{pmatrix}R_{10}(\dot{r}_e^n)_X-R_{00}(\dot{r}_e^n)_Y\\ R_{11}(\dot{r}_e^n)_X-R_{01}(\dot{r}_e^n)_Y\\ R_{12}(\dot{r}_e^n)_X-R_{02}(\dot{r}_e^n)_Y\end{pmatrix}_{\tilde{q}_e^n}\Delta h\end{aligned} \tag{26}$$

2.3 姿勢の運動方程式の線形化

$$\begin{aligned}\frac{d}{dt}(\tilde{q}_n^b+\Delta\tilde{q}_n^b) &= \frac{1}{2}\left\{(\tilde{q}_n^b+\Delta\tilde{q}_n^b)\begin{Bmatrix}0\\ \vec{\omega}_{b/i}^b+\Delta\vec{\omega}_{b/i}^b\end{Bmatrix}\right.\\ &\qquad\left.-\left(\begin{Bmatrix}0\\ \vec{\omega}_{e/i}^n+\Delta\vec{\omega}_{e/i}^n\end{Bmatrix}+\begin{Bmatrix}0\\ \vec{\omega}_{n/e}^n+\Delta\vec{\omega}_{n/e}^n\end{Bmatrix}\right)(\tilde{q}_n^b+\Delta\tilde{q}_n^b)\right\}\\ &\equiv \frac{1}{2}\left\{(\exp(\boldsymbol{u}_n^b)*\tilde{q}_n^b)\begin{Bmatrix}0\\ \vec{\omega}_{b/i}^b+\Delta\vec{\omega}_{b/i}^b\end{Bmatrix}\right.\\ &\qquad\left.-\left(\begin{Bmatrix}0\\ \vec{\omega}_{e/i}^n+\Delta\vec{\omega}_{e/i}^n\end{Bmatrix}+\begin{Bmatrix}0\\ \vec{\omega}_{n/e}^n+\Delta\vec{\omega}_{n/e}^n\end{Bmatrix}\right)(\exp(\boldsymbol{u}_n^b)*\tilde{q}_n^b)\right\}\end{aligned}\tag{27}$$

$$\begin{aligned}\frac{d}{dt}(\tilde{q}_n^b+\Delta\tilde{q}_n^b) &\equiv \frac{d}{dt}(\exp(\boldsymbol{u}_n^b)*\tilde{q}_n^b)\\ &= \frac{d}{dt}\exp(\boldsymbol{u}_n^b)*\tilde{q}_n^b+\exp(\boldsymbol{u}_n^b)*\frac{d}{dt}\tilde{q}_n^b\end{aligned}\tag{28}$$

$$\begin{aligned}\frac{d}{dt}\exp(\boldsymbol{u}_n^b) &= \left(\frac{d}{dt}(\tilde{q}_n^b+\Delta\tilde{q}_n^b)-\exp(\boldsymbol{u}_n^b)*\frac{d}{dt}\tilde{q}_n^b\right)\tilde{q}_n^{b*}\\ &= \left[\frac{1}{2}\left\{(\exp(\boldsymbol{u}_n^b)*\tilde{q}_n^b)\begin{Bmatrix}0\\ \vec{\omega}_{b/i}^b+\Delta\vec{\omega}_{b/i}^b\end{Bmatrix}\right.\right.\\ &\qquad\left.-\left(\begin{Bmatrix}0\\ \vec{\omega}_{e/i}^n+\Delta\vec{\omega}_{e/i}^n\end{Bmatrix}+\begin{Bmatrix}0\\ \vec{\omega}_{n/e}^n+\Delta\vec{\omega}_{n/e}^n\end{Bmatrix}\right)(\exp(\boldsymbol{u}_n^b)*\tilde{q}_n^b)\right\}\\ &\qquad\left.-\exp(\boldsymbol{u}_n^b)\frac{1}{2}\left\{\tilde{q}_n^b\begin{Bmatrix}0\\ \vec{\omega}_{b/i}^b\end{Bmatrix}-\left(\begin{Bmatrix}0\\ \vec{\omega}_{e/i}^n\end{Bmatrix}+\begin{Bmatrix}0\\ \vec{\omega}_{n/e}^n\end{Bmatrix}\right)\tilde{q}_n^b\right\}\right]\tilde{q}_n^{b*}\\ &= \frac{1}{2}\left[(\exp(\boldsymbol{u}_n^b)*\tilde{q}_n^b)\begin{Bmatrix}0\\ \Delta\vec{\omega}_{b/i}^b\end{Bmatrix}\right.\\ &\qquad-\left(\begin{Bmatrix}0\\ \vec{\omega}_{e/i}^n+\Delta\vec{\omega}_{e/i}^n\end{Bmatrix}+\begin{Bmatrix}0\\ \vec{\omega}_{n/e}^n+\Delta\vec{\omega}_{n/e}^n\end{Bmatrix}\right)(\exp(\boldsymbol{u}_n^b)*\tilde{q}_n^b)\\ &\qquad\left.+\exp(\boldsymbol{u}_n^b)\left(\begin{Bmatrix}0\\ \vec{\omega}_{e/i}^n\end{Bmatrix}+\begin{Bmatrix}0\\ \vec{\omega}_{n/e}^n\end{Bmatrix}\right)\tilde{q}_n^b\right]\tilde{q}_n^{b*}\\ &= \frac{1}{2}\left[\exp(\boldsymbol{u}_n^b)\tilde{q}_n^b\begin{Bmatrix}0\\ \Delta\vec{\omega}_{b/i}^b\end{Bmatrix}\tilde{q}_n^{b*}\right.\\ &\qquad-\left(\begin{Bmatrix}0\\ \vec{\omega}_{e/i}^n+\Delta\vec{\omega}_{e/i}^n\end{Bmatrix}+\begin{Bmatrix}0\\ \vec{\omega}_{n/e}^n+\Delta\vec{\omega}_{n/e}^n\end{Bmatrix}\right)\exp(\boldsymbol{u}_n^b)\\ &\qquad\left.+\exp(\boldsymbol{u}_n^b)\left(\begin{Bmatrix}0\\ \vec{\omega}_{e/i}^n\end{Bmatrix}+\begin{Bmatrix}0\\ \vec{\omega}_{n/e}^n\end{Bmatrix}\right)\right]\end{aligned}\tag{29}$$

ここで $|\boldsymbol{u}_n^b| \ll 1$ であるから $\exp(\boldsymbol{u}_e^n) \approx \begin{Bmatrix} 1 \\ \boldsymbol{u}_e^n \end{Bmatrix}$ より

$$
\begin{aligned}
\frac{d}{dt}\exp(\boldsymbol{u}_n^b) \approx \frac{1}{2}\Bigg[& \begin{Bmatrix} 1 \\ \boldsymbol{u}_n^b \end{Bmatrix} \tilde{q}_n^b \begin{Bmatrix} 0 \\ \Delta\vec{\omega}_{b/i}^b \end{Bmatrix} \tilde{q}_n^{b*} \\
& - \left(\begin{Bmatrix} 0 \\ \vec{\omega}_{e/i}^n + \Delta\vec{\omega}_{e/i}^n \end{Bmatrix} + \begin{Bmatrix} 0 \\ \vec{\omega}_{n/e}^n + \Delta\vec{\omega}_{n/e}^n \end{Bmatrix} \right) \begin{Bmatrix} 1 \\ \boldsymbol{u}_n^b \end{Bmatrix} \\
& + \begin{Bmatrix} 1 \\ \boldsymbol{u}_n^b \end{Bmatrix} \left(\begin{Bmatrix} 0 \\ \vec{\omega}_{e/i}^n \end{Bmatrix} + \begin{Bmatrix} 0 \\ \vec{\omega}_{n/e}^n \end{Bmatrix} \right) \Bigg] \\
= \frac{1}{2}\Bigg[& \begin{Bmatrix} 1 \\ \boldsymbol{u}_n^b \end{Bmatrix} \tilde{q}_n^b \begin{Bmatrix} 0 \\ \Delta\vec{\omega}_{b/i}^b \end{Bmatrix} \tilde{q}_n^{b*} \\
& - \begin{Bmatrix} 0 \\ \Delta\vec{\omega}_{e/i}^n + \Delta\vec{\omega}_{n/e}^n + 2(\vec{\omega}_{e/i}^n + \vec{\omega}_{n/e}^n) \times \boldsymbol{u}_n^b \end{Bmatrix} \Bigg]
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
\frac{d}{dt}\boldsymbol{u}_n^b &= \frac{1}{2}\left\{ \mathrm{Rot}\left[\tilde{q}_n^{b*}\right] \Delta\vec{\omega}_{b/i}^b - \Delta\vec{\omega}_{e/i}^n - \Delta\vec{\omega}_{n/e}^n - 2(\vec{\omega}_{e/i}^n + \vec{\omega}_{n/e}^n) \times \boldsymbol{u}_n^b \right\} \\
&= \frac{1}{2} \begin{bmatrix} R_{00} & R_{10} & R_{20} \\ R_{01} & R_{11} & R_{21} \\ R_{02} & R_{12} & R_{22} \end{bmatrix}_{\tilde{q}_n^b} \Delta\vec{\omega}_{b/i}^b - \Omega_{e/i} \begin{bmatrix} R_{01} & -R_{00} & 0 \\ R_{11} & -R_{10} & 0 \\ R_{21} & -R_{20} & 0 \end{bmatrix}_{\tilde{q}_e^n} \boldsymbol{u}_e^n \\
&\quad - \frac{1}{2(r_e+h)} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Delta\dot{\vec{r}}_e^n + \frac{1}{2\,(r_e+h)^2} \begin{pmatrix} (\dot{r}_e^n)_Y \\ -(\dot{r}_e^n)_X \\ 0 \end{pmatrix} \Delta h \\
&\quad - \begin{bmatrix} 0 & -\omega_2 & \omega_1 \\ \omega_2 & 0 & -\omega_0 \\ -\omega_1 & \omega_0 & 0 \end{bmatrix}_{\vec{\omega}_{e/i}^n + \vec{\omega}_{n/e}^n} \boldsymbol{u}_n^b
\end{aligned} \tag{31}
$$

3 Initialize

地球上で静止している状態では

- 速度の運動方程式

$$
\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \tilde{q}_n^b \begin{Bmatrix} 0 \\ \vec{a}^b \end{Bmatrix} \tilde{q}_n^{b*} + \begin{Bmatrix} 0 \\ \vec{g}^n \end{Bmatrix} \tag{32}
$$

- 位置の運動方程式

$$
\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \frac{1}{2}\tilde{q}_e^n \begin{Bmatrix} 0 \\ \vec{\omega}_{n/e}^e \end{Bmatrix} \tag{33}
$$

$$
\because \vec{\omega}_{n/e}^n = 0 \tag{34}
$$

- 姿勢の運動方程式

$$
\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \frac{1}{2}\left[\tilde{q}_n^b \begin{Bmatrix} 0 \\ \vec{\omega}_{b/i}^b \end{Bmatrix} - \begin{Bmatrix} 0 \\ \vec{\omega}_{e/i}^n \end{Bmatrix} \tilde{q}_n^b \right] \tag{35}
$$

以上より

$$
\tilde{q}_n^b \begin{Bmatrix} 0 \\ \vec{a}^b \end{Bmatrix} \tilde{q}_n^{b*} + \begin{Bmatrix} 0 \\ \vec{g}^n \end{Bmatrix} = 0 \tag{36}
$$

$$
\tilde{q}_n^b \begin{Bmatrix} 0 \\ \vec{\omega}_{b/i}^b \end{Bmatrix} = \begin{Bmatrix} 0 \\ \vec{\omega}_{e/i}^n \end{Bmatrix} \tilde{q}_n^b \tag{37}
$$

3.1 速度の運動方程式からの Initialize

初期姿勢を推定する。

$$
\begin{aligned}
0 &= \tilde{q}_n^b \begin{Bmatrix} 0 \\ \vec{a}^b \end{Bmatrix} + \begin{Bmatrix} 0 \\ \vec{g}^n \end{Bmatrix} \tilde{q}_n^b \\
&= \begin{Bmatrix} -\vec{q}\cdot\vec{a} \\ q_0\vec{a}+\vec{q}\times\vec{a} \end{Bmatrix} + \begin{Bmatrix} -\vec{g}\cdot\vec{q} \\ q_0\vec{g}+\vec{g}\times\vec{q} \end{Bmatrix} \\
&= \begin{Bmatrix} -\vec{q}\cdot(\vec{a}+\vec{g}) \\ q_0(\vec{a}+\vec{g})+\vec{q}\times(\vec{a}-\vec{g}) \end{Bmatrix} \\
&= \begin{bmatrix} 0 & -(\vec{a}+\vec{g})_1 & -(\vec{a}+\vec{g})_2 & -(\vec{a}+\vec{g})_3 \\ (\vec{a}+\vec{g})_1 & 0 & (\vec{a}-\vec{g})_3 & -(\vec{a}-\vec{g})_2 \\ (\vec{a}+\vec{g})_2 & -(\vec{a}-\vec{g})_3 & 0 & (\vec{a}-\vec{g})_1 \\ (\vec{a}+\vec{g})_3 & (\vec{a}-\vec{g})_2 & -(\vec{a}-\vec{g})_1 & 0 \end{bmatrix} \begin{bmatrix} q_0 \\ \vec{q} \end{bmatrix}
\end{aligned} \tag{38}
$$

直接的に $\begin{bmatrix} q_0 \\ \vec{q} \end{bmatrix}$ を求めるわけにはいかないので、工夫して iteration する。

- 方法 1

$$
\tilde{q}_{k+1} \equiv \begin{Bmatrix} 1 \\ -\boldsymbol{u} \end{Bmatrix} \tilde{q}_k = \begin{bmatrix} 1 & u_1 & u_2 & u_3 \\ -u_1 & 1 & u_3 & -u_2 \\ -u_2 & -u_3 & 1 & u_1 \\ -u_3 & u_2 & -u_1 & 1 \end{bmatrix} \begin{bmatrix} q_0 \\ \vec{q} \end{bmatrix}_k \tag{39}
$$

代入して

$$
0 = \begin{bmatrix} 0 & -(\vec{a}+\vec{g})_1 & -(\vec{a}+\vec{g})_2 & -(\vec{a}+\vec{g})_3 \\ (\vec{a}+\vec{g})_1 & 0 & (\vec{a}-\vec{g})_3 & -(\vec{a}-\vec{g})_2 \\ (\vec{a}+\vec{g})_2 & -(\vec{a}-\vec{g})_3 & 0 & (\vec{a}-\vec{g})_1 \\ (\vec{a}+\vec{g})_3 & (\vec{a}-\vec{g})_2 & -(\vec{a}-\vec{g})_1 & 0 \end{bmatrix} \begin{bmatrix} 1 & u_1 & u_2 & u_3 \\ -u_1 & 1 & u_3 & -u_2 \\ -u_2 & -u_3 & 1 & u_1 \\ -u_3 & u_2 & -u_1 & 1 \end{bmatrix} \begin{bmatrix} q_0 \\ \vec{q} \end{bmatrix}_k \tag{40}
$$

整理すると

$$
A = B \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{41}
$$

$$
A \equiv - \begin{bmatrix} -q_3(a+g)_3 - q_2(a+g)_2 - q_1(a+g)_1 \\ q_2(a-g)_3 - q_3(a-g)_2 + q_0(a+g)_1 \\ -q_1(a-g)_3 + q_0(a+g)_2 + q_3(a-g)_1 \\ q_0(a+g)_3 + q_1(a-g)_2 - q_2(a-g)_1 \end{bmatrix} \tag{42}
$$

$$
B \equiv \begin{bmatrix} -q_3(a+g)_2+q_2(a+g)_3+q_0(a+g)_1 & q_3(a+g)_1-q_1(a+g)_3+q_0(a+g)_2 & -q_2(a+g)_1+q_1(-a+g)_2+q_0(a+g)_3 \\ q_2(a-g)_2+q_3(a-g)_3+q_1(a+g)_1 & q_2(a+g)_1-q_0(a-g)_3-q_1(a-g)_2 & q_3(a+g)_1+q_0(a-g)_2-q_1(a-g)_3 \\ q_1(a+g)_2-q_0(a-g)_3-q_2(a-g)_1 & q_1(a-g)_1+q_3(a-g)_3+q_2(a+g)_2 & -q_0(a-g)_1+q_3(a+g)_2-q_2(a-g)_3 \\ -q_0(a-g)_2+q_1(a+g)_3-q_3(a-g)_1 & q_0(a-g)_1+q_2(a+g)_3-q_3(a-g)_2 & q_1(a-g)_1+q_2(a-g)_2+q_3(a+g)_3 \end{bmatrix} \tag{43}
$$

最小二乗法を適用して

$$
B^{\mathrm{T}}B \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = B^{\mathrm{T}}A \tag{44}
$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = (B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}A \tag{45}$$

補足

$$A_{4\times1} = B_{4\times3}\boldsymbol{u}_{3\times1} + \boldsymbol{\delta}_{4\times1} \tag{46}$$

$$\boldsymbol{\delta} = A - B\boldsymbol{u} \tag{47}$$

$$\begin{aligned} minimize. \quad \boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{\delta} &= (A - B\boldsymbol{u})^{\mathrm{T}}(A - B\boldsymbol{u}) \\ &= A^{\mathrm{T}}A + \boldsymbol{u}^{\mathrm{T}}B^{\mathrm{T}}B\boldsymbol{u} - A^{\mathrm{T}}B\boldsymbol{u} - \boldsymbol{u}^{\mathrm{T}}B^{\mathrm{T}}A \end{aligned} \tag{48}$$

$$\begin{aligned} then. \quad 0 &= \frac{\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{\delta}}{\partial \boldsymbol{u}^{\mathrm{T}}} \\ &= B^{\mathrm{T}}B\boldsymbol{u} - B^{\mathrm{T}}A \end{aligned} \tag{49}$$

$$\therefore \boldsymbol{u} = (B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}A \tag{50}$$

- 方法 2

  Yawing が 0 固定の場合

$$\tilde{q}_k = \begin{Bmatrix} \cos\frac{\theta}{2}\cos\frac{\phi}{2} \\ \cos\frac{\theta}{2}\sin\frac{\phi}{2} \\ \sin\frac{\theta}{2}\cos\frac{\phi}{2} \\ -\sin\frac{\theta}{2}\sin\frac{\phi}{2} \end{Bmatrix} \tag{51}$$

$$\begin{aligned} \tilde{q}_{k+1} &= \begin{Bmatrix} \cos\frac{\theta+\Delta\theta}{2}\cos\frac{\phi+\Delta\phi}{2} \\ \cos\frac{\theta+\Delta\theta}{2}\sin\frac{\phi+\Delta\phi}{2} \\ \sin\frac{\theta+\Delta\theta}{2}\cos\frac{\phi+\Delta\phi}{2} \\ -\sin\frac{\theta+\Delta\theta}{2}\sin\frac{\phi+\Delta\phi}{2} \end{Bmatrix} \\ &\approx \begin{Bmatrix} \cos\frac{\theta}{2}\cos\frac{\phi}{2} - \sin\frac{\theta}{2}\cos\frac{\phi}{2}\frac{\Delta\theta}{2} - \cos\frac{\theta}{2}\sin\frac{\phi}{2}\frac{\Delta\phi}{2} \\ \cos\frac{\theta}{2}\sin\frac{\phi}{2} - \sin\frac{\theta}{2}\sin\frac{\phi}{2}\frac{\Delta\theta}{2} + \cos\frac{\theta}{2}\cos\frac{\phi}{2}\frac{\Delta\phi}{2} \\ \sin\frac{\theta}{2}\cos\frac{\phi}{2} + \cos\frac{\theta}{2}\cos\frac{\phi}{2}\frac{\Delta\theta}{2} - \sin\frac{\theta}{2}\sin\frac{\phi}{2}\frac{\Delta\phi}{2} \\ -\sin\frac{\theta}{2}\sin\frac{\phi}{2} - \cos\frac{\theta}{2}\sin\frac{\phi}{2}\frac{\Delta\theta}{2} - \sin\frac{\theta}{2}\cos\frac{\phi}{2}\frac{\Delta\phi}{2} \end{Bmatrix} \\ &= \begin{Bmatrix} q_0 - q_2v_1 - q_1v_2 \\ q_1 + q_3v_1 + q_0v_2 \\ q_2 + q_0v_1 + q_3v_2 \\ q_3 - q_1v_1 - q_2v_2 \end{Bmatrix} \end{aligned} \tag{52}$$

$$-\frac{\pi}{2} \leqq \theta \leqq \frac{\pi}{2} \tag{53}$$

$$-\frac{\pi}{2} \leqq \phi \leqq \frac{\pi}{2} \tag{54}$$

$$\sin\frac{\theta}{2} = sign.(q_2)\sqrt{q_2^2 + q_3^2} \tag{55}$$

$$\cos\frac{\theta}{2} = \sqrt{q_0^2 + q_1^2} \tag{56}$$

$$\sin\frac{\phi}{2} = sign.(q_1)\sqrt{q_1^2 + q_3^2} \tag{57}$$

$$\cos\frac{\phi}{2} = \sqrt{q_0^2 + q_2^2} \tag{58}$$

$$0=\begin{bmatrix}0 & -(\vec{a}+\vec{g})_1 & -(\vec{a}+\vec{g})_2 & -(\vec{a}+\vec{g})_3\\(\vec{a}+\vec{g})_1 & 0 & (\vec{a}-\vec{g})_3 & -(\vec{a}-\vec{g})_2\\(\vec{a}+\vec{g})_2 & -(\vec{a}-\vec{g})_3 & 0 & (\vec{a}-\vec{g})_1\\(\vec{a}+\vec{g})_3 & (\vec{a}-\vec{g})_2 & -(\vec{a}-\vec{g})_1 & 0\end{bmatrix}\begin{bmatrix}q_0-q_2v_1-q_1v_2\\q_1+q_3v_1+q_0v_2\\q_2+q_0v_1+q_3v_2\\q_3-q_1v_1-q_2v_2\end{bmatrix}_k \tag{59}$$

$$A=B\begin{bmatrix}v_1\\v_2\end{bmatrix} \tag{60}$$

$$A\equiv-\begin{bmatrix}-(a+g)_1q_1-(a+g)_2q_2-(a+g)_3q_3\\(a+g)_1q_0-(a-g)_2q_3+(a-g)_3q_2\\(a-g)_1q_3+(a+g)_2q_0-(a-g)_3q_1\\-(a-g)_1q_2+(a-g)_2q_1+(a+g)_3q_0\end{bmatrix} \tag{61}$$

$$B\equiv\begin{bmatrix}-(a+g)_1q_3-(a+g)_2q_0+(a+g)_3q_1 & -(a+g)_1q_0-(a+g)_2q_3+(a+g)_3q_2\\-(a+g)_1q_2+(a-g)_2q_1+(a-g)_3q_0 & -(a+g)_1q_1+(a-g)_2q_2+(a-g)_3q_3\\-(a-g)_1q_1-(a+g)_2q_2-(a-g)_3q_3 & -(a-g)_1q_2-(a+g)_2q_1-(a-g)_3q_0\\-(a-g)_1q_0+(a-g)_2q_3-(a+g)_3q_2 & -(a-g)_1q_3+(a-g)_2q_0-(a+g)_3q_1\end{bmatrix} \tag{62}$$

$$\begin{bmatrix}v_1\\v_2\end{bmatrix}=(B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}A \tag{63}$$

3.2 姿勢の運動方程式からの Initialize

4 キャリブレーションについて

4.1 センサの Alignment

$$\begin{cases}a_{11}=K_1\vec{u}_1\cdot\vec{u}_{01}\\a_{12}=K_2\vec{u}_2\cdot\vec{u}_{01}\\a_{13}=K_3\vec{u}_3\cdot\vec{u}_{01}\end{cases} \tag{64}$$

$$\begin{cases}a_{21}=K_1\vec{u}_1\cdot\vec{u}_{02}\\a_{22}=K_2\vec{u}_2\cdot\vec{u}_{02}\\a_{23}=K_3\vec{u}_3\cdot\vec{u}_{02}\end{cases} \tag{65}$$

$$\begin{cases}a_{31}=K_1\vec{u}_1\cdot\vec{u}_{03}\\a_{32}=K_2\vec{u}_2\cdot\vec{u}_{03}\\a_{33}=K_3\vec{u}_3\cdot\vec{u}_{03}\end{cases} \tag{66}$$

$$|\vec{u}_1|=|\vec{u}_2|=|\vec{u}_3|=1 \tag{67}$$

$$\begin{bmatrix}\frac{a_{11}}{K_1} & \frac{a_{12}}{K_2} & \frac{a_{13}}{K_3}\\\frac{a_{21}}{K_1} & \frac{a_{22}}{K_2} & \frac{a_{23}}{K_3}\\\frac{a_{31}}{K_1} & \frac{a_{32}}{K_2} & \frac{a_{33}}{K_3}\end{bmatrix}=\begin{bmatrix}\vec{u}_{01}^{\mathrm{T}}\\\vec{u}_{02}^{\mathrm{T}}\\\vec{u}_{03}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\vec{u}_1 & \vec{u}_2 & \vec{u}_3\end{bmatrix} \tag{68}$$

$$\begin{bmatrix}\vec{u}_1 & \vec{u}_2 & \vec{u}_3\end{bmatrix}=\begin{bmatrix}\vec{u}_{01}^{\mathrm{T}}\\\vec{u}_{02}^{\mathrm{T}}\\\vec{u}_{03}^{\mathrm{T}}\end{bmatrix}^{-1}\begin{bmatrix}\frac{a_{11}}{K_1} & \frac{a_{12}}{K_2} & \frac{a_{13}}{K_3}\\\frac{a_{21}}{K_1} & \frac{a_{22}}{K_2} & \frac{a_{23}}{K_3}\\\frac{a_{31}}{K_1} & \frac{a_{32}}{K_2} & \frac{a_{33}}{K_3}\end{bmatrix} \tag{69}$$

4.2　センサのアライメント補正

Mode1 で使用する ($\vec{u}_{01} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$) ことを考える

$$\begin{cases} v_{\text{output}\,1} = K_1 \vec{v} \cdot \vec{u}_1 \\ v_{\text{output}\,2} = K_2 \vec{v} \cdot \vec{u}_2 \\ v_{\text{output}\,3} = K_3 \vec{v} \cdot \vec{u}_3 \end{cases} \tag{70}$$

ただし、

$$v_{\text{output}} \equiv v_{\text{raw}} - v_{\text{zero}} \tag{71}$$

整理して、

$$\begin{bmatrix} \frac{v_{\text{output}\,1}}{K_1} \\ \frac{v_{\text{output}\,2}}{K_2} \\ \frac{v_{\text{output}\,3}}{K_3} \end{bmatrix} = \begin{bmatrix} \vec{u}_1^{\mathrm{T}} \\ \vec{u}_2^{\mathrm{T}} \\ \vec{u}_3^{\mathrm{T}} \end{bmatrix} \vec{v} \tag{72}$$

よって、

$$\vec{v} = \begin{bmatrix} \vec{u}_1^{\mathrm{T}} \\ \vec{u}_2^{\mathrm{T}} \\ \vec{u}_3^{\mathrm{T}} \end{bmatrix}^{-1} \begin{bmatrix} \frac{v_{\text{output}\,1}}{K_1} \\ \frac{v_{\text{output}\,2}}{K_2} \\ \frac{v_{\text{output}\,3}}{K_3} \end{bmatrix} \tag{73}$$

5　分散について

5.1　Quarternion Error Model と Quarternion Covariance の関係

$$\tilde{q} + \Delta\tilde{q} = \begin{Bmatrix} 1 \\ \vec{u} \end{Bmatrix} \begin{Bmatrix} q_0 \\ \vec{q} \end{Bmatrix} = \begin{Bmatrix} q_0 - \vec{u} \cdot \vec{q} \\ \vec{q} + q_0 \vec{u} + \vec{u} \times \vec{q} \end{Bmatrix} \tag{74}$$

$$\therefore \Delta\tilde{q} = \begin{Bmatrix} -\vec{u} \cdot \vec{q} \\ q_0 \vec{u} + \vec{u} \times \vec{q} \end{Bmatrix} \quad = \begin{bmatrix} -q_1 & -q_2 & -q_3 \\ q_0 & q_3 & -q_2 \\ -q_3 & q_0 & q_1 \\ q_2 & -q_1 & q_0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{75}$$

$$\begin{aligned} \sigma^2(\Delta\tilde{q}) &\equiv E\left[\Delta\tilde{q}(\Delta\tilde{q})^{\mathrm{T}}\right] \\ &= E\left[\begin{bmatrix} -\vec{u} \cdot \vec{q} \\ q_0 \vec{u} + \vec{u} \times \vec{q} \end{bmatrix} \begin{bmatrix} -\vec{u} \cdot \vec{q} \\ q_0 \vec{u} + \vec{u} \times \vec{q} \end{bmatrix}^{\mathrm{T}}\right] \\ &= E\left[\begin{bmatrix} -\vec{u} \cdot \vec{q} \\ q_0 \vec{u} + \vec{u} \times \vec{q} \end{bmatrix} \begin{bmatrix} -\vec{u} \cdot \vec{q} & q_0 \vec{u}^{\mathrm{T}} + (\vec{u} \times \vec{q})^{\mathrm{T}} \end{bmatrix}\right] \\ &= E\left[\begin{bmatrix} (\vec{u} \cdot \vec{q})^2 & -(\vec{u} \cdot \vec{q})(q_0 \vec{u}^{\mathrm{T}} + (\vec{u} \times \vec{q})^{\mathrm{T}}) \\ -(q_0 \vec{u} + \vec{u} \times \vec{q})(\vec{u} \cdot \vec{q}) & q_0^2 \vec{u}\vec{u}^{\mathrm{T}} + q_0 \vec{u}(\vec{u} \times \vec{q})^{\mathrm{T}} + (\vec{u} \times \vec{q}) q_0 \vec{u}^{\mathrm{T}} + (\vec{u} \times \vec{q})(\vec{u} \times \vec{q})^{\mathrm{T}} \end{bmatrix}\right] \\ &= \begin{bmatrix} \vec{q}^{\mathrm{T}} E\left[\vec{u}\vec{u}^{\mathrm{T}}\right] \vec{q} & -q_0 \vec{q}^{\mathrm{T}} E\left[\vec{u}\vec{u}^{\mathrm{T}}\right] + \vec{q}^{\mathrm{T}} E\left[\vec{u}\vec{u}^{\mathrm{T}}\right] \Omega_{\vec{q}}^{\mathrm{T}} \\ -q_0 E\left[\vec{u}\vec{u}^{\mathrm{T}}\right] \vec{q} + \Omega_{\vec{q}} E\left[\vec{u}\vec{u}^{\mathrm{T}}\right] \vec{q} & q_0^2 E\left[\vec{u}\vec{u}^{\mathrm{T}}\right] + q_0 E\left[\vec{u}\vec{u}^{\mathrm{T}}\right] \Omega_{\vec{q}}^{\mathrm{T}} + q_0 \Omega_{\vec{q}} E\left[\vec{u}\vec{u}^{\mathrm{T}}\right] + \Omega_{\vec{q}} E\left[\vec{u}\vec{u}^{\mathrm{T}}\right] \Omega_{\vec{q}}^{\mathrm{T}} \end{bmatrix} \end{aligned} \tag{76}$$

ただし、

$$\Omega_{\vec{q}} \equiv \begin{bmatrix} 0 & \vec{q}_3 & -\vec{q}_2 \\ -\vec{q}_3 & 0 & \vec{q}_1 \\ \vec{q}_2 & -\vec{q}_1 & 0 \end{bmatrix} \tag{77}$$

対角要素だけ欲しい場合は、

$$\begin{cases} \sigma^2(\Delta\tilde{q}_0) \equiv E\left[(\Delta q_0)^2\right] = \vec{q}^{\mathrm{T}} E\left[\vec{u}\vec{u}^{\mathrm{T}}\right]\vec{q} \\ \sigma^2(\Delta\tilde{q}_i) \equiv E\left[(\Delta q_i)^2\right] = \{(q_0 I + \Omega_{\vec{q}}) E\left[\vec{u}\vec{u}^{\mathrm{T}}\right](q_0 I + \Omega_{\vec{q}}^{\mathrm{T}})\}_{i,i} \quad (i = 1 \sim 3) \end{cases} \tag{78}$$

5.2 オイラー角との関係

5.2.1 位置

$$\phi = \arcsin\left\{1 - 2\left((q_e^n)_0{}^2 + (q_e^n)_3{}^2\right)\right\} \tag{79}$$

$$\begin{aligned} \Delta\phi &= \frac{\partial\phi}{\partial(q_e^n)_0}\Delta(q_e^n)_0 + \frac{\partial\phi}{\partial(q_e^n)_1}\Delta(q_e^n)_1 + \frac{\partial\phi}{\partial(q_e^n)_2}\Delta(q_e^n)_2 + \frac{\partial\phi}{\partial(q_e^n)_3}\Delta(q_e^n)_3 \\ &= \frac{-4(q_e^n)_0}{\cos\phi}\Delta(q_e^n)_0 + \frac{-4(q_e^n)_3}{\cos\phi}\Delta(q_e^n)_3 \end{aligned} \tag{80}$$

$$x = \sin y \tag{81}$$

$$\frac{d}{dx}\arcsin x = \frac{d}{dx}y = \frac{dy}{dx}\frac{d}{dy}y = \left(\frac{dx}{dy}\right)^{-1} = \frac{1}{\cos y} \tag{82}$$

$$\lambda = \arctan\frac{(q_e^n)_3}{(q_e^n)_0} - \arctan\frac{(q_e^n)_1}{(q_e^n)_2} \tag{83}$$

$$\begin{aligned} \Delta\lambda &= \frac{\partial\lambda}{\partial(q_e^n)_0}\Delta(q_e^n)_0 + \frac{\partial\lambda}{\partial(q_e^n)_1}\Delta(q_e^n)_1 + \frac{\partial\lambda}{\partial(q_e^n)_2}\Delta(q_e^n)_2 + \frac{\partial\lambda}{\partial(q_e^n)_3}\Delta(q_e^n)_3 \\ &= \frac{-(q_e^n)_3}{(q_e^n)_0{}^2 + (q_e^n)_3{}^2}\Delta(q_e^n)_0 - \frac{(q_e^n)_2}{(q_e^n)_1{}^2 + (q_e^n)_2{}^2}\Delta(q_e^n)_1 \\ &\quad - \frac{-(q_e^n)_1}{(q_e^n)_1{}^2 + (q_e^n)_2{}^2}\Delta(q_e^n)_2 + \frac{(q_e^n)_0}{(q_e^n)_0{}^2 + (q_e^n)_3{}^2}\Delta(q_e^n)_3 \end{aligned} \tag{84}$$

$$\alpha = \arctan\frac{(q_e^n)_3}{(q_e^n)_0} + \arctan\frac{(q_e^n)_1}{(q_e^n)_2} \tag{85}$$

$$\begin{aligned} \Delta\alpha &\equiv 0 \\ &= \frac{\partial\alpha}{\partial(q_e^n)_0}\Delta(q_e^n)_0 + \frac{\partial\alpha}{\partial(q_e^n)_1}\Delta(q_e^n)_1 + \frac{\partial\alpha}{\partial(q_e^n)_2}\Delta(q_e^n)_2 + \frac{\partial\alpha}{\partial(q_e^n)_3}\Delta(q_e^n)_3 \\ &= \frac{-(q_e^n)_3}{(q_e^n)_0{}^2 + (q_e^n)_3{}^2}\Delta(q_e^n)_0 + \frac{(q_e^n)_2}{(q_e^n)_1{}^2 + (q_e^n)_2{}^2}\Delta(q_e^n)_1 \\ &\quad + \frac{-(q_e^n)_1}{(q_e^n)_1{}^2 + (q_e^n)_2{}^2}\Delta(q_e^n)_2 + \frac{(q_e^n)_0}{(q_e^n)_0{}^2 + (q_e^n)_3{}^2}\Delta(q_e^n)_3 \end{aligned} \tag{86}$$

また

$$1 \equiv ((q_e^n)_0 + \Delta(q_e^n)_0)^2 + ((q_e^n)_1 + \Delta(q_e^n)_1)^2 + ((q_e^n)_2 + \Delta(q_e^n)_2)^2 + ((q_e^n)_3 + \Delta(q_e^n)_3)^2 \tag{87}$$

ゆえに

$$0 \equiv (q_e^n)_0 \Delta(q_e^n)_0 + (q_e^n)_1 \Delta(q_e^n)_1 + (q_e^n)_2 \Delta(q_e^n)_2 + (q_e^n)_3 \Delta(q_e^n)_3 \tag{88}$$

以上まとめると

$$\begin{bmatrix} \cos\phi\Delta\phi \\ \Delta\lambda \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -4(q_e^n)_0 & 0 & 0 & -4(q_e^n)_3 \\ -\frac{(q_e^n)_3}{(q_e^n)_0{}^2+(q_e^n)_3{}^2} & -\frac{(q_e^n)_2}{(q_e^n)_1{}^2+(q_e^n)_2{}^2} & \frac{(q_e^n)_1}{(q_e^n)_1{}^2+(q_e^n)_2{}^2} & \frac{(q_e^n)_0}{(q_e^n)_0{}^2+(q_e^n)_3{}^2} \\ -\frac{(q_e^n)_3}{(q_e^n)_0{}^2+(q_e^n)_3{}^2} & \frac{(q_e^n)_2}{(q_e^n)_1{}^2+(q_e^n)_2{}^2} & -\frac{(q_e^n)_1}{(q_e^n)_1{}^2+(q_e^n)_2{}^2} & \frac{(q_e^n)_0}{(q_e^n)_0{}^2+(q_e^n)_3{}^2} \\ (q_e^n)_0 & (q_e^n)_1 & (q_e^n)_2 & (q_e^n)_3 \end{bmatrix} \begin{bmatrix} \Delta(q_e^n)_0 \\ \Delta(q_e^n)_1 \\ \Delta(q_e^n)_2 \\ \Delta(q_e^n)_3 \end{bmatrix} \tag{89}$$

$\Delta\lambda$ を求めるだけなら

$$\lambda + \alpha = 2\arctan\frac{(q_e^n)_3}{(q_e^n)_0} \tag{90}$$

であるから

$$\begin{aligned} \Delta\lambda &= \Delta(\lambda+\alpha) \quad (\because \Delta\alpha \equiv 0) \\ &= \Delta\left(2\arctan\frac{(q_e^n)_3}{(q_e^n)_0}\right) \\ &= 2\left(\frac{-(q_e^n)_3}{(q_e^n)_0{}^2+(q_e^n)_3{}^2}\Delta(q_e^n)_0 + \frac{(q_e^n)_0}{(q_e^n)_0{}^2+(q_e^n)_3{}^2}\Delta(q_e^n)_3\right) \end{aligned} \tag{91}$$

まとめると

$$\begin{bmatrix} \cos\phi\Delta\phi \\ \left((q_e^n)_0{}^2+(q_e^n)_3{}^2\right)\Delta\lambda \end{bmatrix} = \begin{bmatrix} -4(q_e^n)_0 & 0 & 0 & -4(q_e^n)_3 \\ -2(q_e^n)_3 & 0 & 0 & 2(q_e^n)_0 \end{bmatrix} \begin{bmatrix} \Delta(q_e^n)_0 \\ \Delta(q_e^n)_1 \\ \Delta(q_e^n)_2 \\ \Delta(q_e^n)_3 \end{bmatrix} \tag{92}$$

整理して

$$\begin{bmatrix} \frac{\cos\phi}{-4}\Delta\phi \\ \frac{\left((q_e^n)_0{}^2+(q_e^n)_3{}^2\right)}{-2}\Delta\lambda \end{bmatrix} = \begin{bmatrix} (q_e^n)_0 & (q_e^n)_3 \\ (q_e^n)_3 & -(q_e^n)_0 \end{bmatrix} \begin{bmatrix} \Delta(q_e^n)_0 \\ \Delta(q_e^n)_3 \end{bmatrix} \tag{93}$$

$$\begin{bmatrix} \Delta(q_e^n)_0 \\ \Delta(q_e^n)_3 \end{bmatrix} = \frac{1}{-(q_e^n)_0{}^2-(q_e^n)_3{}^2} \begin{bmatrix} (q_e^n)_0 & -(q_e^n)_3 \\ -(q_e^n)_3 & -(q_e^n)_0 \end{bmatrix} \begin{bmatrix} \frac{\cos\phi}{-4}\Delta\phi \\ \frac{\left((q_e^n)_0{}^2+(q_e^n)_3{}^2\right)}{-2}\Delta\lambda \end{bmatrix} \tag{94}$$

さらに

$$\begin{bmatrix} \cos\phi\Delta\phi \\ \left((q_e^n)_0{}^2+(q_e^n)_3{}^2\right)\Delta\lambda \end{bmatrix} = \begin{bmatrix} -4(q_e^n)_0 & 0 & 0 & -4(q_e^n)_3 \\ -2(q_e^n)_3 & 0 & 0 & 2(q_e^n)_0 \end{bmatrix} \begin{bmatrix} -q_1 & -q_2 & -q_3 \\ q_0 & q_3 & -q_2 \\ -q_3 & q_0 & q_1 \\ q_2 & -q_1 & q_0 \end{bmatrix}_{\tilde{q}_e^n} \begin{bmatrix} \Delta u_1 \\ \Delta u_2 \\ \Delta u_3 \end{bmatrix} \tag{95}$$

5.2.2 姿勢

$$\begin{aligned}\Psi &= \arctan \frac{2\left((q_n^b)_1 (q_n^b)_2 + (q_n^b)_0 (q_n^b)_3\right)}{(q_n^b)_0{}^2 + (q_n^b)_1{}^2 - (q_n^b)_2{}^2 - (q_n^b)_3{}^2} \\ &= \arctan \frac{2\left((q_n^b)_1 (q_n^b)_2 + (q_n^b)_0 (q_n^b)_3\right)}{1 - 2\left((q_n^b)_2{}^2 + (q_n^b)_3{}^2\right)} \\ &= \arctan \frac{2\left((q_n^b)_1 (q_n^b)_2 + (q_n^b)_0 (q_n^b)_3\right)}{2\left((q_n^b)_0{}^2 + (q_n^b)_1{}^2\right) - 1}\end{aligned} \tag{96}$$

$$\begin{aligned}\Delta\Psi &= \frac{\partial \Psi}{\partial (q_n^b)_0}\Delta(q_n^b)_0 + \frac{\partial \Psi}{\partial (q_n^b)_1}\Delta(q_n^b)_1 + \frac{\partial \Psi}{\partial (q_n^b)_2}\Delta(q_n^b)_2 + \frac{\partial \Psi}{\partial (q_n^b)_3}\Delta(q_n^b)_3 \\ &= \left(\frac{1}{1+\tan^2\Psi}\right)\left(\frac{2(q_n^b)_3}{denom._\Psi}\Delta(q_n^b)_0 + \frac{2(q_n^b)_2}{denom._\Psi}\Delta(q_n^b)_1 + \frac{2(q_n^b)_1}{denom._\Psi}\Delta(q_n^b)_2 + \frac{2(q_n^b)_0}{denom._\Psi}\Delta(q_n^b)_3\right)\end{aligned} \tag{97}$$

$$denom._\Psi \equiv (q_n^b)_0{}^2 + (q_n^b)_1{}^2 - (q_n^b)_2{}^2 - (q_n^b)_3{}^2 \tag{98}$$

$$y = \arctan x \tag{99}$$

$$\frac{d}{dx}\arctan x = \frac{d}{dx} y = \left(\frac{dx}{dy}\right)^{-1} = \cos^2 y = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2} \tag{100}$$

$$\Theta = \arcsin\left(2\left((q_n^b)_0 (q_n^b)_2 - (q_n^b)_1 (q_n^b)_3\right)\right) \tag{101}$$

$$\begin{aligned}\Delta\Theta &= \frac{\partial \Theta}{\partial (q_n^b)_0}\Delta(q_n^b)_0 + \frac{\partial \Theta}{\partial (q_n^b)_1}\Delta(q_n^b)_1 + \frac{\partial \Theta}{\partial (q_n^b)_2}\Delta(q_n^b)_2 + \frac{\partial \Theta}{\partial (q_n^b)_3}\Delta(q_n^b)_3 \\ &= \frac{2(q_n^b)_2}{\cos\Theta}\Delta(q_n^b)_0 + \frac{-2(q_n^b)_3}{\cos\Theta}\Delta(q_n^b)_1 + \frac{2(q_n^b)_0}{\cos\Theta}\Delta(q_n^b)_2 + \frac{-2(q_n^b)_1}{\cos\Theta}\Delta(q_n^b)_3\end{aligned} \tag{102}$$

$$\Phi = \arctan \frac{2\left((q_n^b)_2 (q_n^b)_3 + (q_n^b)_0 (q_n^b)_1\right)}{(q_n^b)_0{}^2 - (q_n^b)_1{}^2 - (q_n^b)_2{}^2 + (q_n^b)_3{}^2} \tag{103}$$

$$\begin{aligned}\Delta\Phi &= \frac{\partial \Phi}{\partial (q_n^b)_0}\Delta(q_n^b)_0 + \frac{\partial \Phi}{\partial (q_n^b)_1}\Delta(q_n^b)_1 + \frac{\partial \Phi}{\partial (q_n^b)_2}\Delta(q_n^b)_2 + \frac{\partial \Phi}{\partial (q_n^b)_3}\Delta(q_n^b)_3 \\ &= \left(\frac{1}{1+\tan^2\Phi}\right)\left(\frac{2(q_n^b)_1}{denom._\Phi}\Delta(q_n^b)_0 + \frac{2(q_n^b)_0}{denom._\Phi}\Delta(q_n^b)_1 + \frac{2(q_n^b)_3}{denom._\Phi}\Delta(q_n^b)_2 + \frac{2(q_n^b)_2}{denom._\Phi}\Delta(q_n^b)_3\right)\end{aligned} \tag{104}$$

$$denom._\Phi \equiv (q_n^b)_0{}^2 - (q_n^b)_1{}^2 - (q_n^b)_2{}^2 + (q_n^b)_3{}^2 \tag{105}$$

また

$$1 \equiv \left((q_n^b)_0 + \Delta(q_n^b)_0\right)^2 + \left((q_n^b)_1 + \Delta(q_n^b)_1\right)^2 + \left((q_n^b)_2 + \Delta(q_n^b)_2\right)^2 + \left((q_n^b)_3 + \Delta(q_n^b)_3\right)^2 \tag{106}$$

ゆえに

$$0 \equiv (q_n^b)_0 \Delta(q_n^b)_0 + (q_n^b)_1 \Delta(q_n^b)_1 + (q_n^b)_2 \Delta(q_n^b)_2 + (q_n^b)_3 \Delta(q_n^b)_3 \tag{107}$$

以上まとめて

$$
\begin{bmatrix} \frac{1+\tan^2\Psi}{denom._\Psi}\Delta\Psi \\ \cos\Theta\Delta\Theta \\ \frac{1+\tan^2\Phi}{denom._\Phi}\Delta\Phi \\ 0 \end{bmatrix}
= \begin{bmatrix} 2(q_n^b)_3 & 2(q_n^b)_2 & 2(q_n^b)_1 & 2(q_n^b)_0 \\ 2(q_n^b)_2 & -2(q_n^b)_3 & 2(q_n^b)_0 & -2(q_n^b)_1 \\ 2(q_n^b)_1 & 2(q_n^b)_0 & 2(q_n^b)_3 & 2(q_n^b)_2 \\ (q_n^b)_0 & (q_n^b)_1 & (q_n^b)_2 & (q_n^b)_3 \end{bmatrix}
\begin{bmatrix} \Delta(q_n^b)_0 \\ \Delta(q_n^b)_1 \\ \Delta(q_n^b)_2 \\ \Delta(q_n^b)_3 \end{bmatrix} \tag{108}
$$

さらに

$$
\begin{bmatrix} \frac{1+\tan^2\Psi}{denom._\Psi}\Delta\Psi \\ \cos\Theta\Delta\Theta \\ \frac{1+\tan^2\Phi}{denom._\Phi}\Delta\Phi \\ 0 \end{bmatrix}
= \begin{bmatrix} 2(q_n^b)_3 & 2(q_n^b)_2 & 2(q_n^b)_1 & 2(q_n^b)_0 \\ 2(q_n^b)_2 & -2(q_n^b)_3 & 2(q_n^b)_0 & -2(q_n^b)_1 \\ 2(q_n^b)_1 & 2(q_n^b)_0 & 2(q_n^b)_3 & 2(q_n^b)_2 \\ (q_n^b)_0 & (q_n^b)_1 & (q_n^b)_2 & (q_n^b)_3 \end{bmatrix}
\begin{bmatrix} -q_1 & -q_2 & -q_3 \\ q_0 & q_3 & -q_2 \\ -q_3 & q_0 & q_1 \\ q_2 & -q_1 & q_0 \end{bmatrix}_{\tilde{q}_n^b}
\begin{bmatrix} \Delta u_1 \\ \Delta u_2 \\ \Delta u_3 \end{bmatrix} \tag{109}
$$

6 ドリフト除去に関して

6.1 エラーモデルを仮定する場合

1 自由度モデルでのドリフト除去

運動方程式

$$\theta_{t+1} = \theta_t + (\omega_t - \omega_{0t})\Delta t$$

$$\hat{\theta}_{t+1} = \hat{\theta}_t + (\hat{\omega}_t - \hat{\omega}_{0t})\Delta t$$

$$\Delta\theta_{t+1} = \Delta\theta_t + (\underbrace{\Delta\omega}_{e_1} - \Delta\omega_{0t})\Delta t$$

バイアス変動

$$\omega_{0t+1} = (1-\beta\Delta t)\omega_{0t} + \underbrace{N\sqrt{\Delta t}u}_{e_2}$$

$$\hat{\omega}_{0t+1} = (1-\beta\Delta t)\hat{\omega}_{0t}$$

$$\Delta\omega_{0t+1} = (1-\beta\Delta t)\Delta\omega_0 + e_2$$

以上あわせて

$$
\begin{bmatrix} \Delta\theta \\ \Delta\omega_0 \end{bmatrix}_{t+1} = \begin{bmatrix} 1 & -\Delta t \\ 0 & 1-\beta\Delta t \end{bmatrix} \begin{bmatrix} \Delta\theta \\ \Delta\omega_0 \end{bmatrix}_t + \begin{bmatrix} \Delta t & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}
$$

6.2　平均値推定

$$\underline{x}_{k+1} = A_k \underline{x}_k + B_k \underline{w} \tag{110}$$

$$\underline{z}_k = H_k \underline{x}_k + \underline{v} \tag{111}$$

$$E[\underline{w}] = \underline{\mu} \neq \underline{0} \tag{112}$$

$$V[\underline{w}] \equiv E[(\underline{w} - \underline{\mu})(\underline{w} - \underline{\mu})^{\mathrm{T}}] = Q \tag{113}$$

$$E[\underline{v}] = 0 \tag{114}$$

$$V[\underline{v}] \equiv E[(\underline{v} - 0)(\underline{v} - 0)^{\mathrm{T}}] = R \tag{115}$$

$$E[\underline{w}_{ideal}] \equiv \underline{0} \tag{116}$$

$$V[\underline{w}_{ideal}] \equiv V[w] = Q \tag{117}$$

$$\underline{x}_{ideal\ k+1} = A_k \underline{x}_k + B_k \underline{w}_{ideal} \tag{118}$$

$$\begin{aligned} \underline{x}_{k+1} &= A_k \underline{x}_k + B_k \underline{w} \\ &\equiv A_k \underline{x}_k + B_k \underline{\mu}_k + B_k \underline{w}_{ideal} \end{aligned} \tag{119}$$

$$P_{k+1} = A_k P_k A_k^{\mathrm{T}} + B_k Q B_k^{\mathrm{T}} \tag{120}$$

$$P_{k+1} = A_k P_k A_k^{\mathrm{T}} + B_k Q B_k^{\mathrm{T}} \tag{121}$$

$$\hat{\underline{x}}_{ideal\ k+1} = \underline{x}_{ideal\ k+1} + K\left[\underline{z}_{k+1} - H\underline{x}_{ideal\ k+1}\right] \tag{122}$$

$$\begin{aligned} \hat{\underline{x}}_{k+1} &= \underline{x}_{k+1} + K\left[\underline{z}_{k+1} - H_{k+1}\underline{x}_{k+1}\right] \\ &= \underline{x}_{k+1} + K\left[\underline{z}_{k+1} - H_{k+1}\underline{x}_{ideal\ k} - H_{k+1}B_k\underline{\mu}_k\right] \end{aligned} \tag{123}$$

$$K_{k+1} = P_{k+1}H_{k+1}^{\mathrm{T}}\left(H_{k+1}P_{k+1}H_{k+1}^{\mathrm{T}} + R_{k+1}\right)^{-1} \tag{124}$$

$$K_{k+1} = P_{k+1}H_{k+1}^{\mathrm{T}}\left(H_{k+1}P_{k+1}H_{k+1}^{\mathrm{T}} + R_{k+1}\right)^{-1} \tag{125}$$

$$\hat{P}_{k+1} = [I - K_{k+1}H_{k+1}]P_{k+1} \tag{126}$$

$$\hat{P}_{k+1} = [I - K_{k+1}H_{k+1}]P_{k+1} \tag{127}$$

以上より

$$\begin{aligned} E\left[z_{k+1} - H_{k+1}x_{k+1}\right] &= E\left[z_{k+1} - H_{k+1}x_{ideal\ k+1} - H_{k+1}B_k\underline{\mu}_k\right] \\ &= E\left[z_{k+1} - H_{k+1}x_{ideal\ k+1}\right] - E\left[H_{k+1}B_k\underline{\mu}_k\right] \\ &= 0 - E\left[H_{k+1}B_k\underline{\mu}_k\right] \\ &= -E\left[H_{k+1}B_k\underline{\mu}_k\right] \\ &= -\left(H_{k+1}B_k\right)E\left[\underline{\mu}_k\right] \end{aligned} \tag{128}$$

$$
\begin{aligned}
R &\equiv V\left[z_{k+1} - H_{k+1} x_{k+1}\right] \\
&= V\left[z_{k+1} - H_{k+1} x_{ideal\,k+1} - H_{k+1} B_k \underline{\mu}_k\right] \\
&= V\left[z_{k+1} - H_{k+1} x_{ideal\,k+1}\right] + V\left[H_{k+1} B_k \underline{\mu}_k\right] \\
&= V\left[z_{k+1} - H_{k+1} x_{ideal\,k+1}\right]
\end{aligned}
\tag{129}
$$

平均値除去について考えてみる

$$
\left(H_{k+1} B_k\right)^{\mathrm{T}} E\left[z_{k+1} - H_{k+1} x_{k+1}\right] = -\left(H_{k+1} B_k\right)^{\mathrm{T}} \left(H_{k+1} B_k\right) E\left[\mu_k\right] \tag{130}
$$

最小二乗法より

$$
E\left[\mu_k\right] = -\left(\left(H_{k+1} B_k\right)^{\mathrm{T}} \left(H_{k+1} B_k\right)\right)^{-1} \left(H_{k+1} B_k\right) E\left[z_{k+1} - H_{k+1} x_{k+1}\right] \tag{131}
$$

$$
\mu_{k+1} = \mu_k - E\left[\mu_k\right] \tag{132}
$$

7　オイラー角を用いた場合

運動方程式は以下のとおり。

$$\frac{d\phi}{dt} = -\cos\alpha(\vec{\omega}^n_{n/e})_Y - \sin\alpha(\vec{\omega}^n_{n/e})_X \tag{133}$$

$$\begin{aligned}
\frac{d}{dt}\Delta\phi &= \sin\alpha(\vec{\omega}^n_{n/e})_Y\Delta\alpha - \cos\alpha\Delta(\vec{\omega}^n_{n/e})_Y - \cos\alpha(\vec{\omega}^n_{n/e})_X\Delta\alpha - \sin\alpha\Delta(\vec{\omega}^n_{n/e})_X \\
&= \left(\sin\alpha(\vec{\omega}^n_{n/e})_Y - \cos\alpha(\vec{\omega}^n_{n/e})_X\right)\Delta\alpha \\
&\quad + \frac{\cos\alpha}{r_e+h}\Delta(\dot{r}^n_e)_X - \frac{(\dot{r}^n_e)_X\cos\alpha}{(r_e+h)^2}\Delta h - \frac{\sin\alpha}{r_e+h}\Delta(\dot{r}^n_e)_Y + \frac{(\dot{r}^n_e)_Y\sin\alpha}{(r_e+h)^2}\Delta h \\
&= \left(\sin\alpha(\vec{\omega}^n_{n/e})_Y - \cos\alpha(\vec{\omega}^n_{n/e})_X\right)\Delta\alpha \\
&\quad + \frac{\cos\alpha}{r_e+h}\Delta(\dot{r}^n_e)_X - \frac{\sin\alpha}{r_e+h}\Delta(\dot{r}^n_e)_Y + \left(-\frac{(\dot{r}^n_e)_X\cos\alpha}{(r_e+h)^2} + \frac{(\dot{r}^n_e)_Y\sin\alpha}{(r_e+h)^2}\right)\Delta h
\end{aligned} \tag{134}$$

$$\frac{d\lambda}{dt} = \frac{-\sin\alpha(\vec{\omega}^n_{n/e})_Y + \cos\alpha(\vec{\omega}^n_{n/e})_X}{\cos\phi} \tag{135}$$

$$\begin{aligned}
\frac{d}{dt}\Delta\lambda &= \frac{-\cos\alpha(\vec{\omega}^n_{n/e})_Y\Delta\alpha - \sin\alpha\Delta(\vec{\omega}^n_{n/e})_Y - \sin\alpha(\vec{\omega}^n_{n/e})_X\Delta\alpha + \cos\alpha\Delta(\vec{\omega}^n_{n/e})_X}{\cos\phi} \\
&\quad + \frac{-(\dot{\lambda}\cos\phi)\sin\phi}{\cos\phi^2}\Delta\phi \\
&= \frac{-\cos\alpha(\vec{\omega}^n_{n/e})_Y\Delta\alpha - \sin\alpha\Delta(\vec{\omega}^n_{n/e})_Y - \sin\alpha(\vec{\omega}^n_{n/e})_X\Delta\alpha + \cos\alpha\Delta(\vec{\omega}^n_{n/e})_X}{\cos\phi} \\
&\quad - \dot{\lambda}\tan\phi\Delta\phi \\
&= \frac{-\cos\alpha(\vec{\omega}^n_{n/e})_Y - \sin\alpha(\vec{\omega}^n_{n/e})_X}{\cos\phi}\Delta\alpha \\
&\quad - \frac{\sin\alpha}{\cos\phi}\left(-\frac{1}{r_e+h}\Delta(\dot{r}^n_e)_X + \frac{(\dot{r}^n_e)_X}{(r_e+h)^2}\Delta h\right) \\
&\quad + \frac{\cos\alpha}{\cos\phi}\left(\frac{1}{r_e+h}\Delta(\dot{r}^n_e)_Y - \frac{(\dot{r}^n_e)_Y}{(r_e+h)^2}\Delta h\right) \\
&\quad - \dot{\lambda}\tan\phi\Delta\phi \\
&= \frac{-\cos\alpha(\vec{\omega}^n_{n/e})_Y - \sin\alpha(\vec{\omega}^n_{n/e})_X}{\cos\phi}\Delta\alpha \\
&\quad + \frac{\sin\alpha}{(r_e+h)\cos\phi}\Delta(\dot{r}^n_e)_X + \frac{\cos\alpha}{(r_e+h)\cos\phi}\Delta(\dot{r}^n_e)_Y \\
&\quad - \frac{\sin\alpha(\dot{r}^n_e)_X + \cos\alpha(\dot{r}^n_e)_Y}{(r_e+h)^2\cos\phi}\Delta h \\
&\quad - \dot{\lambda}\tan\phi\Delta\phi
\end{aligned} \tag{136}$$

$$\frac{d\alpha}{dt} = \dot{\lambda}\sin\phi \tag{137}$$

$$\frac{d}{dt}\Delta\alpha = \sin\phi\Delta\dot{\lambda} + \dot{\lambda}\cos\phi\Delta\phi \tag{138}$$

$$\Delta\vec{\omega}^n_{n/e} \approx \frac{1}{r_e+h}\begin{pmatrix}\Delta(\dot{r}^n_e)_Y \\ -\Delta(\dot{r}^n_e)_X \\ 0\end{pmatrix} - \frac{1}{(r_e+h)^2}\begin{pmatrix}(\dot{r}^n_e)_Y \\ -(\dot{r}^n_e)_X \\ 0\end{pmatrix}\Delta h \tag{139}$$

$$\Delta\rho_X \equiv \Delta(\vec{\omega}^n_{n/e})_X = \frac{1}{r_e+h}\Delta(\dot{r}^n_e)_Y - \frac{(\dot{r}^n_e)_Y}{(r_e+h)^2}\Delta h \tag{140}$$

$$\Delta\rho_Y \equiv \Delta(\vec{\omega}^n_{n/e})_Y = -\frac{1}{r_e+h}\Delta(\dot{r}^n_e)_X + \frac{(\dot{r}^n_e)_X}{(r_e+h)^2}\Delta h \tag{141}$$

$$\Delta\rho_Z \equiv \Delta(\vec{\omega}^n_{n/e})_Z = 0 \tag{142}$$

$$\mathrm{DCM}[\phi,\lambda,\alpha] = \begin{bmatrix} -\cos\alpha\sin\phi\cos\lambda - \sin\alpha\sin\lambda & -\cos\alpha\sin\phi\sin\lambda + \sin\alpha\cos\lambda & \cos\alpha\cos\phi \\ \sin\alpha\sin\phi\cos\lambda - \cos\alpha\sin\lambda & \sin\alpha\sin\phi\sin\lambda + \cos\alpha\cos\lambda & -\sin\alpha\cos\phi \\ -\cos\phi\cos\lambda & -\cos\phi\sin\lambda & -\sin\phi \end{bmatrix} \tag{143}$$

$$\begin{aligned}
\Delta\mathrm{DCM}[\phi,\lambda,\alpha] &= \left[\begin{matrix} c\alpha s\phi s\lambda\Delta\lambda - c\alpha c\phi c\lambda\Delta\phi + s\alpha s\phi c\lambda\Delta\alpha - s\alpha c\lambda\Delta\lambda - c\alpha s\lambda\Delta\alpha \\ -s\alpha s\phi s\lambda\Delta\lambda + s\alpha c\phi c\lambda\Delta\phi + c\alpha s\phi c\lambda\Delta\alpha - c\alpha c\lambda\Delta\lambda + s\alpha s\lambda\Delta\alpha \\ c\phi s\lambda\Delta\lambda + s\phi c\lambda\Delta\phi \end{matrix}\right. \\
&\qquad \begin{matrix} -c\alpha s\phi c\lambda\Delta\lambda - c\alpha s\phi s\lambda\Delta\phi + s\alpha s\phi s\lambda\Delta\alpha - s\alpha s\lambda\Delta\lambda + c\alpha c\lambda\Delta\alpha \\ s\alpha s\phi c\lambda\Delta\lambda + s\alpha c\phi s\lambda\Delta\phi + c\alpha s\phi s\lambda\Delta\alpha - c\alpha s\lambda\Delta\lambda - s\alpha c\lambda\Delta\alpha \\ -c\phi c\lambda\Delta\lambda + s\phi s\lambda\Delta\phi \end{matrix} \\
&\qquad \left.\begin{matrix} -c\alpha s\phi\Delta\phi - s\alpha c\phi\Delta\alpha \\ s\alpha s\phi\Delta\phi - c\alpha c\phi\Delta\alpha \\ -c\phi\Delta\phi \end{matrix}\right] \\
&= \left[\begin{matrix} -c\alpha c\phi c\lambda\Delta\phi + (c\alpha s\phi s\lambda - s\alpha c\lambda)\Delta\lambda + (s\alpha s\phi c\lambda - c\alpha s\lambda)\Delta\alpha \\ s\alpha c\phi c\lambda\Delta\phi - (s\alpha s\phi s\lambda + c\alpha c\lambda)\Delta\lambda + (c\alpha s\phi c\lambda + s\alpha s\lambda)\Delta\alpha \\ s\phi c\lambda\Delta\phi + c\phi s\lambda\Delta\lambda \end{matrix}\right. \\
&\qquad \begin{matrix} -c\alpha s\phi s\lambda\Delta\phi - (c\alpha s\phi c\lambda + s\alpha s\lambda)\Delta\lambda + (s\alpha s\phi s\lambda + c\alpha c\lambda)\Delta\alpha \\ s\alpha c\phi s\lambda\Delta\phi + (s\alpha s\phi c\lambda - c\alpha s\lambda)\Delta\lambda + (c\alpha s\phi s\lambda - s\alpha c\lambda)\Delta\alpha \\ s\phi s\lambda\Delta\phi - c\phi c\lambda\Delta\lambda \end{matrix} \\
&\qquad \left.\begin{matrix} -c\alpha s\phi\Delta\phi - s\alpha c\phi\Delta\alpha \\ s\alpha s\phi\Delta\phi - c\alpha c\phi\Delta\alpha \\ -c\phi\Delta\phi \end{matrix}\right] \\
&= \begin{bmatrix} -c\alpha c\phi c\lambda & -c\alpha s\phi s\lambda & -c\alpha s\phi \\ s\alpha c\phi c\lambda & s\alpha c\phi s\lambda & s\alpha s\phi \\ s\phi c\lambda & s\phi s\lambda & -c\phi \end{bmatrix}\Delta\phi \\
&\quad + \begin{bmatrix} (c\alpha s\phi s\lambda - s\alpha c\lambda) & (c\alpha s\phi c\lambda + s\alpha s\lambda) & 0 \\ -(s\alpha s\phi s\lambda + c\alpha c\lambda) & (s\alpha s\phi c\lambda - c\alpha s\lambda) & 0 \\ c\phi s\lambda & -c\phi c\lambda & 0 \end{bmatrix}\Delta\lambda \\
&\quad + \begin{bmatrix} (s\alpha s\phi c\lambda - c\alpha s\lambda) & (s\alpha s\phi s\lambda + c\alpha c\lambda) & -s\alpha c\phi \\ (c\alpha s\phi c\lambda + s\alpha s\lambda) & (c\alpha s\phi s\lambda - s\alpha c\lambda) & -c\alpha c\phi \\ 0 & 0 & 0 \end{bmatrix}\Delta\alpha
\end{aligned} \tag{144}$$

$$\begin{aligned}
\vec{\omega}^n_{e/i} &= \mathrm{DCM}[\phi,\lambda,\alpha]\,\vec{\omega}^i_{e/i} \\
&= \mathrm{DCM}[\phi,\lambda,\alpha]\begin{pmatrix}0 \\ 0 \\ \Omega_{e/i}\end{pmatrix}
\end{aligned} \tag{145}$$

$$
\begin{aligned}
\Delta\vec{\omega}^{n}_{e/i} &= \Delta\mathrm{DCM}\left[\phi,\lambda,\alpha\right]\begin{pmatrix}0\\0\\\Omega_{e/i}\end{pmatrix}\\
&= \Omega_{e/i}\begin{pmatrix}-c\alpha s\phi\Delta\phi - s\alpha c\phi\Delta\alpha\\ s\alpha s\phi\Delta\phi - c\alpha c\phi\Delta\alpha\\ -c\phi\Delta\phi\end{pmatrix}\\
&= \Omega e/i\begin{bmatrix}-c\alpha s\phi & 0 & -s\alpha c\phi\\ s\alpha s\phi & 0 & -c\alpha c\phi\\ -c\phi & 0 & 0\end{bmatrix}\begin{bmatrix}\Delta\phi\\\Delta\lambda\\\Delta\alpha\end{bmatrix}
\end{aligned}
\tag{146}
$$

$$
\begin{aligned}
\frac{d}{dt}\begin{Bmatrix}0\\\dot{\vec{r}}^{n}_{e}\end{Bmatrix} = \tilde{q}^{n*}_{b}\begin{Bmatrix}0\\\vec{a}^{b}\end{Bmatrix}\tilde{q}^{n}_{b} + \begin{Bmatrix}0\\\vec{g}^{n}\end{Bmatrix}\\
-\begin{Bmatrix}0\\\left(2\vec{\omega}^{n}_{e/i}+\vec{\omega}^{n}_{n/e}\right)\times\dot{\vec{r}}^{n}_{e}\end{Bmatrix} - \mathrm{DCM}\left[\phi,\lambda,\alpha\right]\begin{Bmatrix}0\\\vec{\omega}^{e}_{e/i}\times\left(\vec{\omega}^{e}_{e/i}\times\vec{r}_{e}\right)\end{Bmatrix}
\end{aligned}
\tag{147}
$$

$$
\begin{aligned}
\frac{d}{dt}\Delta\dot{\vec{r}}^{n}_{e} = \mathrm{Rot}\left[\tilde{q}^{b*}_{n}\right]\Delta\vec{a}^{b} - 2\mathrm{Rot}\left[\tilde{q}^{b*}_{n}\right]\vec{a}^{b}\times\Delta\vec{u}^{b}_{n} + \Delta\vec{g}^{n} + \dot{\vec{r}}^{n}_{e}\times\left(2\Delta\vec{\omega}^{n}_{e/i}+\Delta\vec{\omega}^{n}_{n/e}\right) - \left(2\vec{\omega}^{n}_{e/i}+\vec{\omega}^{n}_{n/e}\right)\times\Delta\dot{\vec{r}}^{n}_{e}\\
-\Delta\mathrm{DCM}\left[\phi,\lambda,\alpha\right]\left(\vec{\omega}^{e}_{e/i}\times\left(\vec{\omega}^{e}_{e/i}\times\vec{r}_{e}\right)\right) - \mathrm{DCM}\left[\phi,\lambda,\alpha\right]\Delta\left(\vec{\omega}^{e}_{e/i}\times\left(\vec{\omega}^{e}_{e/i}\times\vec{r}_{e}\right)\right)
\end{aligned}
\tag{148}
$$

$$
\begin{aligned}
\dot{\vec{r}}^{n}_{e}\times 2\Delta\vec{\omega}^{n}_{e/i} &= 2\Omega_{e/i}\begin{bmatrix}0 & -(\dot{\vec{r}}^{n}_{e})_{Z} & (\dot{\vec{r}}^{n}_{e})_{X}\\ (\dot{\vec{r}}^{n}_{e})_{Z} & 0 & -(\dot{\vec{r}}^{n}_{e})_{X}\\ -(\dot{\vec{r}}^{n}_{e})_{Y} & (\dot{\vec{r}}^{n}_{e})_{X} & 0\end{bmatrix}\begin{bmatrix}-c\alpha s\phi & 0 & -s\alpha c\phi\\ s\alpha s\phi & 0 & -c\alpha c\phi\\ -c\phi & 0 & 0\end{bmatrix}\begin{bmatrix}\Delta\phi\\\Delta\lambda\\\Delta\alpha\end{bmatrix}\\
&= 2\Omega_{e/i}\begin{bmatrix}-s\alpha s\phi(\dot{\vec{r}}^{n}_{e})_{Z} - c\phi(\dot{\vec{r}}^{n}_{e})_{Y} & 0 & c\alpha c\phi(\dot{\vec{r}}^{n}_{e})_{Z}\\ -c\alpha s\phi(\dot{\vec{r}}^{n}_{e})_{Z} + c\phi(\dot{\vec{r}}^{n}_{e})_{X} & 0 & -s\alpha c\phi(\dot{\vec{r}}^{n}_{e})_{Z}\\ c\alpha s\phi(\dot{\vec{r}}^{n}_{e})_{Y} + s\alpha s\phi(\dot{\vec{r}}^{n}_{e})_{X} & 0 & s\alpha c\phi(\dot{\vec{r}}^{n}_{e})_{Y} - c\alpha c\phi(\dot{\vec{r}}^{n}_{e})_{Y}\end{bmatrix}\begin{bmatrix}\Delta\phi\\\Delta\lambda\\\Delta\alpha\end{bmatrix}
\end{aligned}
\tag{149}
$$

$$
\begin{aligned}
\dot{\vec{r}}^{n}_{e}\times\Delta\vec{\omega}^{n}_{n/e} &= \begin{bmatrix}0 & -(\dot{\vec{r}}^{n}_{e})_{Z} & (\dot{\vec{r}}^{n}_{e})_{Y}\\ (\dot{\vec{r}}^{n}_{e})_{Z} & 0 & -(\dot{\vec{r}}^{n}_{e})_{X}\\ -(\dot{\vec{r}}^{n}_{e})_{Y} & (\dot{\vec{r}}^{n}_{e})_{X} & 0\end{bmatrix}\begin{bmatrix}\Delta\rho_{X}\\\Delta\rho_{Y}\\\Delta\rho_{Z}\end{bmatrix}\\
&= \begin{bmatrix}-(\dot{\vec{r}}^{n}_{e})_{Z}\Delta\rho_{Y}\\ (\dot{\vec{r}}^{n}_{e})_{Z}\Delta\rho_{X}\\ -(\dot{\vec{r}}^{n}_{e})_{Y}\Delta\rho_{X} + (\dot{\vec{r}}^{n}_{e})_{X}\Delta\rho_{Y}\end{bmatrix}\\
&= \begin{bmatrix}\frac{(\dot{\vec{r}}^{n}_{e})_{Z}}{r_e+h}\Delta(\dot{\vec{r}}^{n}_{e})_{X} - \frac{(\dot{\vec{r}}^{n}_{e})_{X}(\dot{\vec{r}}^{n}_{e})_{Z}}{(r_e+h)^2}\Delta h\\ \frac{(\dot{\vec{r}}^{n}_{e})_{Z}}{r_e+h}\Delta(\dot{\vec{r}}^{n}_{e})_{Y} - \frac{(\dot{\vec{r}}^{n}_{e})_{Y}(\dot{\vec{r}}^{n}_{e})_{Z}}{(r_e+h)^2}\Delta h\\ -\frac{(\dot{\vec{r}}^{n}_{e})_{X}}{r_e+h}\Delta(\dot{\vec{r}}^{n}_{e})_{X} - \frac{(\dot{\vec{r}}^{n}_{e})_{Y}}{r_e+h}\Delta(\dot{\vec{r}}^{n}_{e})_{Y} + \frac{{(\dot{\vec{r}}^{n}_{e})_{X}}^2+{(\dot{\vec{r}}^{n}_{e})_{Y}}^2}{(r_e+h)^2}\Delta h\end{bmatrix}
\end{aligned}
\tag{150}
$$

$$
\vec{\omega}^{e}_{e/i}\times\left(\vec{\omega}^{e}_{e/i}\times\vec{r}_{e}\right) = {\Omega_{e/i}}^{2}(R_{\mathrm{normal}}+h)\cos\phi\begin{pmatrix}-\cos\lambda\\-\sin\lambda\\0\end{pmatrix}
\tag{151}
$$

$$
\begin{aligned}
\Delta\left(\vec{\omega}^{e}_{e/i}\times\left(\vec{\omega}^{e}_{e/i}\times\vec{r}_{e}\right)\right) &= {\Omega_{e/i}}^{2}\Delta\left((R_{\text{normal}}+h)\cos\phi\begin{pmatrix}-\cos\lambda\\-\sin\lambda\\0\end{pmatrix}\right)\\
&= {\Omega_{e/i}}^{2}\cos\phi\begin{pmatrix}-\cos\lambda\\-\sin\lambda\\0\end{pmatrix}\Delta h\\
&\quad -{\Omega_{e/i}}^{2}(R_{\text{normal}}+h)\sin\phi\begin{pmatrix}-\cos\lambda\\-\sin\lambda\\0\end{pmatrix}\Delta\phi\\
&\quad +{\Omega_{e/i}}^{2}(R_{\text{normal}}+h)\cos\phi\begin{pmatrix}\sin\lambda\\-\cos\lambda\\0\end{pmatrix}\Delta\lambda
\end{aligned}
\tag{152}
$$

$$
\frac{d}{dt}\Delta\vec{u}^{b}_{n} = \frac{1}{2}\left\{\text{Rot}\left[\tilde{q}^{b*}_{n}\right]\Delta\vec{\omega}^{b}_{b/i}-\Delta\vec{\omega}^{n}_{e/i}-\Delta\vec{\omega}^{n}_{n/e}-2(\vec{\omega}^{n}_{e/i}+\vec{\omega}^{n}_{n/e})\times\Delta\vec{u}^{b}_{n}\right\}
\tag{153}
$$